ERROR PATTERNS IN COMPUTATION

Using Error Patterns to Improve Instruction

NINTH EDITION

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Preface

We want to help our students learn to think mathematically. Mere skill in computing is not our primary concern.

Even so, computation in its many forms—estimation, mental computation, with calculators and computers, and using paper-and-pencil procedures—continues to have a significant role in both the learning of mathematics and in solving problems within the world around us. This is made clear in Principles and Standards for School Mathematics, published by the National Council of Teachers of Mathematics (NCTM) in 2000.

We hope all of our students will understand what they study; we want them to succeed in mathematics and to enjoy solving problems—but students sometimes learn misconceptions. As they learn about operations and methods of computation, they sometimes adopt erroneous procedures in spite of our best efforts.

This book was written for those who are willing to listen carefully to what each student says and to make thoughtful analyses of student work—teachers who want to help each student by discovering error patterns they may be using; and to be enabled, thereby, to focus instruction more effectively. This book was written to help teachers, whatever the level, to look at all student work diagnostically.

Error Patterns in Computation has changed greatly through the years. The first 98 pages of this edition are devoted to instructional issues; they provide help with diagnosing misconceptions and error patterns in computation; and they present many ideas for teaching varied methods of computation.

Much of the book focuses on detecting the systematic errors many students make when computing with paper and pencil. Reasons students may have learned erroneous procedures are considered, and strategies for helping those students are presented. Of course, many of the instructional strategies described are useful when teaching any student—whether that student has experienced difficulty under previous instruction or not.

The erroneous patterns displayed by students are not due to carelessness alone nor are they due to insufficient practice. Students make connections, observe patterns, and make inferences during instruction. In the case of an error pattern,
that which has been learned does not always produce correct answers. Interestingly, some incorrect procedures produce correct answers part of the time; and when this happens students are reinforced in their belief that they have learned the desired concepts and skills.

You will find looking for error patterns to be a very worthwhile assessment activity. You will gain more specific knowledge of each student’s strengths upon which to base future instruction. Whenever you observe error patterns in your own classroom, be sure to refrain from assigning practice activities that reinforce incorrect concepts and procedures.

**NEW TO THIS EDITION**

As with the previous edition, this ninth edition reflects many of the concerns of NCTM’s *Principles and Standards for School Mathematics*, including emphasis on concepts as well as skills. Read the statements in the following list. Some discuss changes to this edition; some provide new directives worth noting about this edition.

In this edition:

- There is a focus on computational fluency. Early in Chapter 1 the section titled “Computational Fluency” discusses computational fluency and the need for students to flexibly use varied methods of computation depending on the problem situation.
- Computation, by whatever method, is distinguished from the operation itself. In Chapter 3, separate sections address meanings of operations and various methods of computation.
- Understanding the meanings of operations is emphasized. In Chapter 3 the section titled “Helping Students Understand Operations” demonstrates the value of total-part-part meanings in helping students know when to use each operation—whatever method of computation may be appropriate.
- There is great emphasis on helping students understand the big ideas. In Chapter 3 the section titled “Helping Students Understand Big Ideas” addresses important ideas that help students achieve success with mathematics: meanings of numerals, equals, and equivalence; properties of operations; and compensation principles.
- Different methods of computation are presented with an emphasis on using each method when appropriate. Specific methods are presented in different sections of Chapter 3; paper-and-pencil procedures are considered in the context of other methods of computation. Figure 1.1 provides help for deciding which form of computation is needed.
- Varied means of assessment are described and illustrated; self-assessment is emphasized. Additional material on interviewing is included.
Appendix E, new to this edition, describes a procedure for planning, conducting, and reporting a diagnostic interview. A rubric is provided.

There are two new sections in Chapter 3: “Working with Parents” as well as “Students with Learning Disabilities.”

The annotated list of Selected Resources, with sections on Assessment and Diagnosis and on Instruction, has been thoroughly updated.

ORGANIZATION OF THE TEXT

The book is organized into two parts with appendices. Part One considers the place of computation within our age of calculators and computers, then focuses on various aspects of assessment and instruction—for both concepts and skills. In Part Two, sample student papers are presented in chapters that focus on a particular mathematical topic. Within each chapter you have opportunities to identify patterns of error and to reflect on corrective instruction, then compare your ideas with those of the author.

Experience has shown that direct involvement through simulation, as provided in this text, helps both preservice and inservice teachers become more proficient at diagnosing and correcting computational procedures. You gain skill by actually looking for patterns, making decisions, and planning instruction.

Additional student papers are included in Appendix A where you have further opportunities to practice identifying error patterns.

ACKNOWLEDGMENTS

I wish to express appreciation for the encouragement of many classroom teachers who have shown great interest in this book over the years, and to acknowledge the help of teachers, former students, and their students. These colleagues have identified many of the error patterns presented.

I also wish to thank my colleagues who served as text reviewers. I sincerely appreciate revision suggestions made by Nancy Alexander, Louisiana Tech University; Mary Margaret Capraro, Texas A&M; Patricia Hewitt, The University of Tennessee; Fredrick L. Silverman, University of Northern Colorado; and Steven W. Ziebarth, Western Michigan University.

—Robert B. Ashlock
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This book is designed to help us improve mathematics instruction in our classrooms by becoming more diagnostically oriented. Diagnosis should be continuous throughout instruction.

Although much of this book focuses on computation, many of the ideas presented in Part One can be applied throughout the curriculum if we want to become more sensitive and responsive to where our students are in their development.

How important is it to teach paper-and-pencil computation procedures in our age of calculators and computers? Chapter 1 addresses this concern, and the need for conceptually oriented instruction. Paper-and-pencil procedures are only one of the methods of computation people use when solving problems; we need to make sure our students are able to use them all and know when each is most appropriate.

Why do our students sometimes learn misconceptions and erroneous procedures when learning to compute? Chapter 2 examines the learning process and presents helpful diagnostic concepts and procedures. Tools for diagnostic teaching are also described. Diagnostic teaching involves careful observation; it attempts to determine the concepts that individuals are truly learning and the procedures they are really employing—whether correct or not.

As we learn more about a particular student, we tailor our instruction to that student; we observe and learn more and adjust instruction again. We repeat this diagnosis-instruction cycle as often as necessary. In Chapter 3, there are many ideas to help us design instruction to meet the needs of each student.
In this age of calculators and computers, do our students actually need to learn paper-and-pencil procedures? We want our students to understand mathematical concepts and to compute fluently, but how does this relate to students learning to do paper-and-pencil procedures when calculators are so readily available?

As we examine these and other questions in this chapter, we will find that even in our technological age paper-and-pencil computation is often needed. True, paper-and-pencil procedures constitute only one alternative for computing—though it often makes sense to use such procedures. It is also true that while our students are learning to compute with paper-and-pencil, their knowledge of number combinations and numeration concepts can be further developed—knowledge needed for using calculators and for doing other forms of computation.

**INSTRUCTION IN MATHEMATICS**

Our society is drenched with data. We have long recognized that verbal literacy is essential to our well being as a society; now we recognize that quantitative literacy or *numeracy* is also essential.
Accordingly, our goals are changing. We want to see instructional programs enable students to understand and use mathematics in a technological world. We are not interested in students just doing arithmetic in classrooms; we want to see the operations of arithmetic applied in real-world contexts where students observe and organize data. We no longer assume that students must be skillful with computation before they can actually begin investigating interesting topics in mathematics.

Instruction in mathematics is moving toward covering fewer topics but in greater depth and toward making connections between mathematical ideas. Increasingly, mathematics is being perceived as a science of patterns rather than a collection of rules. In truth, there are those who characterize algebra as generalized arithmetic, and there are those who even propose that “... the teaching and learning of arithmetic be conceived as the foundation for algebra.”\(^2\) The computations of arithmetic are not being ignored. The importance of computation is made clear in *Principles and Standards for School Mathematics*, published in 2000 by the National Council of Teachers of Mathematics (NCTM).

Knowing basic number combinations—the single-digit addition and multiplication pairs and their counterparts for subtraction and division—is essential. Equally essential is computational fluency—having and using efficient and accurate methods for computing.\(^3\)

Number and Operations is one of the five content standards for Grades Pre-K through 12 in *Principles and Standards for School Mathematics*. Number combinations and computation are often involved when the other four content standards—Algebra, Geometry, Measurement, and Data Analysis and Probability—are learned and applied. Moreover, application in every grade of the five process standards—Problem Solving, Reasoning and Proof, Communication, Connections, and Representation—frequently entails number combinations and computation. Number combinations and computation are very much a part of standards-based instruction in mathematics today.

**Computational Fluency**

Increasingly, we need to integrate arithmetic with the world of our students, including their experiences with other subject areas. In order to solve problems encountered in the world around them, our students need to know not only how to compute a needed number, but also when to compute. In order for them to know when to use specific operations, we need to emphasize the meanings of operations during instruction.

If students are to gain computational fluency, they need to learn different methods of computation to use in varied problem-solving situations. The following quotation from NCTM’s *Principles and Standards for School Mathematics* emphasizes this point in relation to computation.
Part of being able to compute fluently means making smart choices about which tools to use and when. Students should have experiences that help them learn to choose among mental computation, paper-and-pencil strategies, estimation, and calculator use. The particular context, the question, and the numbers involved all play roles in those choices.4

If we focus on paper-and-pencil procedures but do not introduce other methods of computing, our students are apt to believe that the process of computing is limited to paper-and-pencil procedures.

When students have a problem to solve, there are decisions to be made before any required computation begins. Consider this part of a conversation overheard in a student math group led by Chi. Calculators were available and students were free to use one if they needed it, but the teacher also encouraged other methods of exact computation: mental computation and paper-and-pencil procedures.

CHI: We have three word problems to solve. Here’s Problem A. You are to help the class get ready for art class. There are 45 pounds of clay for 20 people. How many pounds do you give each person?
RAUL: We need an exact answer. Each person should get exactly the same amount.
TERRY: I don’t think we need to use the calculator. Each person gets a little more than two pounds . . . but how much more?
SONJA: That’s easy, just divide 45 by 20. The students proceed to divide 45 by 20 with paper and pencil.
CHI: Here’s Problem B. Wanda’s scores for three games of darts are 18, 27, and 39. What is her average score?
RAUL: Another exact answer.
CHI: Shall we get the calculator?
SONJA: I can do it in my head.
TERRY: I don’t see how. I’m going to use paper; it’s easy.
SONJA: Round each number up . . . 20 + 30 + 40 is 90 . . . divided by 3 is 30. But we need to subtract: 2 and 3 (that’s 5) and one more . . . 6 divided by 3 is 2. Two from 30 is 28. Her average score is 28.
TERRY: That’s what I got, too.

Different methods of computation are listed in Figure 1.1. Because an approximation is often sufficient, the first decision a student must make is whether an exact

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<th>METHODS OF COMPUTATION</th>
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<td><strong>Approximation</strong></td>
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<td><strong>Exact Computation</strong></td>
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<td>– Mental computation</td>
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<td>– Calculator</td>
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<td>– Paper-and-pencil</td>
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</table>
number is needed. In regard to exact computation, paper-and-pencil procedures constitute only one of the methods of computation available.

The diagram in Figure 1.2 also focuses on decisions about the method of computation to be used for solving a particular problem. (Note that estimation should be involved even when an exact answer is needed.) As we teach computation, we must help our students learn when each particular method of computation is appropriate. The actual teaching of different methods of computation is addressed in Chapter 3.

When a solution to \(300 - 25 = \) is needed by a fourth grader, mental computation is likely the most appropriate method to use. But there are times when a paper-and-pencil strategy is the most efficient procedure for an individual. When a calculator is not immediately available and an exact answer is needed for the sum
of two or three multi-digit numbers, it often makes sense to use a paper-and-pencil procedure.

Because computational procedures are tools for helping us solve problems, whenever possible the context for teaching different methods of computation should be a problem-solving situation; we need to keep focused on problem solving as we teach computation procedures.

The goal of instruction in computation today continues to be computational fluency. Students need to be able to use efficient and accurate methods for computing if they are to enjoy success in most areas of mathematics. Moreover, while studying the operations of arithmetic and various procedures, our students need to experience meaningful learning—to question and understand how mathematical concepts are applied in varied contexts.

ALGORITHMS

It is common to speak of algorithmic thinking, which uses specific step-by-step procedures, in contrast to thinking which is more self-referential and recursive. Polya’s four-step model for problem solving is an example of algorithmic thinking.

An algorithm is a step-by-step procedure for accomplishing a task, such as solving a problem. In this book, the term usually refers to paper-and-pencil procedures for finding a sum, difference, product, or quotient. The paper-and-pencil procedures that individuals learn and use differ over time and among cultures. If a “standard” algorithm is included in your curriculum, remember that curriculum designers made a choice. If some students have already learned different algorithms (for example, a different way to subtract learned in Mexico or in Europe), remember that these students’ procedures are quite acceptable if they always provide the correct number.

Usiskin lists several reasons for teaching various types of algorithms, a few of which follow. These apply to paper-and-pencil procedures as well as to the other types of algorithms he discusses.

- **Power.** An algorithm applies to a class of problems (e.g., multiplication with fractions).
- **Reliability and accuracy.** Done correctly, an algorithm always provides the correct answer.
- **Speed.** An algorithm proceeds directly to the answer.
- **A record.** A paper-and-pencil algorithm provides a record of how the answer was determined.
- **Instruction.** Numeration concepts and properties of operations are applied.

CONCEPTUAL LEARNING AND PROCEDURAL LEARNING

The importance of conceptual learning is stressed by NCTM in *Principles and Standards for School Mathematics*. Conceptual learning in mathematics always focuses
on ideas and on generalizations that make connections among ideas. In contrast, procedural learning focuses on skills and step-by-step procedures.

Sadly, procedures are sometimes taught without adequately connecting the steps to mathematical ideas. Both conceptual learning and procedural learning are necessary, but procedural learning needs to be tied to conceptual learning and to real life applications. Procedural learning must be based on concepts already learned. There is evidence that learning rote procedures before learning concepts and how they are applied in those procedures actually interferes with later meaningful learning.

In order for concepts to build on one another, ideas need to be understood and woven together. As a part of their increasing number sense, our younger students need to understand principles and concepts associated with whole numbers and numerals for whole numbers. Then, students begin to make reasonable estimates and accurate mental computations.

Students need to understand the meaning of each operation (and not just do the computations), so they can decide which operation is needed in particular situations. Otherwise, they do not know which button to push on a calculator or which paper-and-pencil procedure to use.

Conceptual understanding is so important that some mathematics educators stress the invention of algorithms by young students; they fear that early introduction of standard algorithms may be detrimental and not lead to understanding important concepts. Understanding the concepts and reasoning involved in an algorithm does lead to a more secure mastery of that procedure. It is also true that standard algorithms can be taught so that students understand the concepts and reasoning associated with specific procedures.

Paper-and-pencil procedures that we teach actually involve more than procedural knowledge; they entail conceptual knowledge as well. Many of the instructional activities described in Chapter 3 and beyond in this book are included because students need to understand specific concepts. Our students are not merely mechanical processors; they are involved conceptually as they learn—even when we teach procedures.

instruction can emphasize conceptual understanding without sacrificing skill proficiency understanding does not detract from skill proficiency and may even enhance it.

It has long been recognized that instruction should balance conceptual understanding and skill proficiency. One of the classic publications of mathematics education is William Brownell’s “Meaning and Skill—Maintaining the Balance,” published originally in 1956 but reprinted twice by the National Council of Teachers of Mathematics, once as recently as 2003. It must be recognized that as a student uses a specific paper-and-pencil procedure over time, it becomes more automatic. The student employs increasingly less conceptual knowledge and more procedural knowledge, a process researchers sometimes call “proceduralization.”
ERROR PATTERNS IN COMPUTATION

Errors are a positive thing in the process of learning—or at least they should be. Some view errors as part of the "messiness" of doing mathematics.14 In many cultures, errors are regarded as an opportunity to reflect and learn.

Rather than warning our students about errors to avoid, we can use errors as catalysts for learning by approaching errors as problem-solving situations. For example, a group of students can examine an erroneous procedure and use reasoning with concepts they know to determine why the strategy that was used does not always produce a correct answer.15

As they learn computation procedures, many students—even students who invent their own algorithms—learn error patterns. Chapters 4 through 12 include specific examples of error patterns—systematic procedures that students learn but which most often do not provide the correct answer. Sometimes error patterns do produce the correct answer; when they do, students (and often teachers) assume that a correct procedure has been learned.

Algorithms incorporating error patterns are sometimes called buggy algorithms. A buggy algorithm includes at least one erroneous step, and the procedure does not consistently accomplish the intended purpose.

As we teach computation procedures, we need to remember that our students are not necessarily learning what we think we are teaching; we need to keep our eyes and ears open to find out what our students are actually learning. We need to be alert for error patterns!

PAPER-AND-PENCIL PROCEDURES TODAY

Some mathematics educators emphasize a need for greater skill with the procedures of arithmetic. For example, Tom Loveless and John Coughlan, writing in Educational Leadership in 2004, state reasons they believe computation skills are important:

- They are necessary to advance in mathematics and the sciences.
- They are an increasingly important predictor of adult earnings.
- They promote equity in math achievement.16

Loveless and Coughlan conclude:

We would simply like all students to learn how to add, subtract, multiply, and divide using whole numbers, fractions, and decimals—and accurately compute percentages—by the end of 8th grade. Only by mastering these skills will students have the opportunity to learn higher-level mathematics.17

NCTM’s Principles and Standards for School Mathematics clearly supports teaching computation skills:

... students must become fluent in arithmetic computation—they must have efficient and accurate methods that are supported by an understanding of
numbers and operations. “Standard” algorithms for arithmetic computation are one means of achieving this fluency.\(^8\)

The required arithmetic computation skills include estimation, mental computation, and use of calculators as well as paper-and-pencil procedures. But it is noteworthy that even in an age of calculators and computers, students need to be able to use appropriate paper-and-pencil algorithms when it makes sense to do so.

As we teach our students to use paper-and-pencil algorithms, we need to remember that they sometimes learn error patterns. We need to be diagnosticians, carefully observing what our students do and looking for patterns in their written work. The next chapter is designed to help us approach instruction diagnostically.

**REFERENCES**

17. Ibid, p. 58.
Chapter 2

Diagnosing Misconceptions and Error Patterns in Computation

I cannot teach students well if I do not know them well.

Theodore Sizer

Arithmetic is where the answer is right and everything is nice and you can look out of the window and see the blue sky—or the answer is wrong and you have to start all over and try again and see how it comes out this time.

Carl Sandburg

As Sizer reminds us, before we can teach our students, we need to know them well. Learning is a very personal process.

The student into whose mind Sandburg leads us seems to view arithmetic and possibly all of mathematics as an either-or sort of thing. The answer is correct and arithmetic is enjoyable and life is rosy, or the answer is not correct and arithmetic and life are frustrating. We may wonder why this student and others are so answer oriented. Yet, we do need to face the question of why some students do not seem to be able to get the correct answers they need.

This chapter is designed to help you find out. As you read, you will learn why students sometimes learn misconceptions and incorrect procedures. But misconceptions and error patterns should not be the only focus of formative assessment. We must get at the thinking of students by also collecting evidence regarding concepts understood and skills attained; then we can plan instruction that builds on each student's prior attainments. This is emphasized in the Assessment Principle of NCTM's Principles and Standards for School Mathematics (2000).

To maximize the instructional value of assessment, teachers need to move beyond a superficial “right or wrong” analysis of tasks to a focus on how students are thinking about the tasks. Efforts should be made to identify
valuable student insights on which further progress can be based rather than to concentrate solely on errors or misconceptions.³

In this chapter, you will discover how to encourage self-assessment and read about tools you can use for gathering data and making inferences. An extensive discussion of interviews is included. Specific suggestions for creating diagnostic questions and tasks are given, as well as principles which should guide diagnosis.

We expect all students to learn, and for many of our students, we will need to go beyond careful examination of their work. We will need to probe deeply for information about their learning.

### The Equity Principle

Excellence in mathematics education requires equity—high expectations and strong support for all students.⁴

### The Assessment Principle

Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.⁵

As we teach mathematics, we need to be continually assessing—gathering and using information about student learning. Assessment is one of six principles for school mathematics recognized by the National Council of Teachers of Mathematics.

Assessment is the process of gathering information about student learning and the use of that information to plan instruction. Assessment must be aligned with curricular goals; even specific assessment tasks must be planned with these goals in mind.

Though we need to be continually assessing, we do not need to be continually testing. We should also collect evidences of student learning from projects and from various writings and tasks—many of which are described in this chapter. By gathering data from a variety of sources, we are more likely to obtain an accurate picture of each student’s knowledge, skills, and learning processes.

Some educators believe that we as teachers do not know how to teach, or even what specifics to teach, unless we know how we will assess it—the particular evidences of learning we will collect.

Many sources of data are simply called “student work.” We must examine student work carefully and note the different strategies for computing which students have developed. Of course, we observe that results are correct or incorrect; but we also need to look for evidences that indicate how each student is thinking. One way of getting at that thinking is to encourage students to show or describe how they
A helpful practice is for groups of teachers to examine student work together, thereby making examination of student work a collaborative experience.

The purposes of both assessment and diagnosis are to improve learning performance. The processes are similar: collect evidences in relation to curricular goals, make inferences, and plan instruction. However, the term assessment is often used broadly; for example, with reference to a curriculum standard. Diagnosis, the term used frequently in this book, typically has a more narrow focus. Evidences collected as a part of a diagnosis usually focus on a limited number of specific concepts or skills.

Consider Fred’s paper (Figure 2.2). If we merely determine how many answers are correct and how many are incorrect, we will not learn why his answers are not correct. Examine Fred’s paper and note that when multiplication involves renaming, his answer is often incorrect. Look for a pattern among the incorrect responses; observe that he seems to be adding his “crutch” and then multiplying. This can be verified by studying other examples and briefly interviewing Fred.

Because we observed Fred’s error pattern, instruction can be modified as needed. The algorithm may be reviewed as a written record of multiplying “one part at a time” (an application of distributing multiplication over addition), or a

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FIGURE 2.1 Six students’ solutions to $25 + 37$.

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<tr>
<th>Student 1</th>
<th>Student 2</th>
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<tr>
<td>See I know thirty&lt;br&gt; + twenty = fifty&lt;br&gt; seven + five = 12&lt;br&gt; fifty + 12 = 62.</td>
<td>25 + 37&lt;br&gt; 62</td>
<td>$25 + \frac{37}{2}$&lt;br&gt; $i3 \ 02$</td>
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<th>Student 4</th>
<th>Student 5</th>
<th>Student 6</th>
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| 25 I added the<br> 5 and 7<br> together that is 12 so I carried the 1<br> and put down the 2. 1 2 3 = 6. I put down the 6 so it | 25 2 + 3 = 5 and<br> $+37 8 + 7 = 12$
53 12 $rac{53}{12}$ I add two to the 6 and made 53 | $25 \frac{5 + 7}{12}$
is 13 $+37 3 \ 8$
I added $\frac{37}{8}$ is 5 |


obtained their answers, as can be seen for six students’ solutions to $25 + 37$ in Figure 2.1.
FIGURE 2.2 Sample student paper.

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<td>2.</td>
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<td>x2</td>
<td>x4</td>
<td>x6</td>
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<td>86</td>
<td>124</td>
<td>308</td>
</tr>
</tbody>
</table>

4. | 2 | 5 |
---|---|---|
| 35 | 63 | 58 |
| X5 | X7 | X6 |
| 255 | 561 | 548 |

modification of the algorithm itself may be suggested so that the “crutch” is recorded with a small half-space digit written below the line. (See Error Pattern M-W-2 in Chapter 5.)

Rather than just scoring papers, we need to examine each student’s paper diagnostically—looking for patterns, hypothesizing possible causes, and verifying our ideas. As we learn about each student, we will find that a student’s paper is sometimes a problem or puzzle to be solved. Researchers have known for a long time that we can learn much by carefully looking for evidence of misconceptions and by observing erroneous procedures.

LEARNING MISCONCEPTIONS AND ERROR PATTERNS

Error patterns reveal misconceptions that have been learned, but how are they learned? The mathematical ideas and procedures (or rules) a student learns may be correct or they may be full of misconceptions, but the process of learning those ideas and procedures is basically the same. During experiences with a concept or a process (or a procedure), a student focuses on whatever the experiences appear to have in common and connects that information to information already known.
Consider the student whose only school experiences with the number idea we call *five* involve manila cards with black dots in the familiar domino pattern (Figure 2.3a). That student may draw from those experiences a notion of five that includes many or all of the characteristics his experiences had in common: possibly black on manila paper, round dots, or a specific configuration. One of the author's own students, when presenting to her students the configuration associated with Stern pattern boards (Figure 2.3b) was told, “That's not five. Five doesn't look like that.”

More young students will name as a triangle the shape in Figure 2.3c than the shape in Figure 2.3d; yet both are triangles. Again, configuration (or even the orientation of the figure) may be a common characteristic of a child's limited range of experiences with triangles.

Dr. Geoffrey Matthews, who organized the Nuffield Mathematics Teaching Project in England, told about a child who computed correctly one year but missed half of the problems the next year. As the child learned to compute, he adopted the rule, “Begin on the side by the piano.” The next school year the child was in a room with the piano on the other side, and he was understandably confused about where to start computing.

Concept cards, which are often used in learning centers, also illustrate concept formation (Figure 2.4). As a student examines a concept card, he sees a name or label, such as *rhombus*. Examples of a rhombus and non-examples of a rhombus are both shown on the card, and the student must decide what a rhombus is. Finally, the card provides an opportunity to test out his newly derived definition. Students often learn erroneous concepts and processes similarly. They look for commonalities among their initial contacts with an idea or procedure. They form an abstraction with certain common characteristics, and their concept or
algorithm is formed. The common attributes may be very specific, such as crossing out a digit, placing a digit in front of another, or finding the difference between two one-digit numbers (regardless of order). Failure to consider enough examples is one of the errors of inductive learning often cited by those who study thinking.

Because our students connect new information with what they already know, it is very important that we assess the preconceptions of our students. Prior knowledge is not always correct knowledge; misconceptions are common. Even when our students correctly observe particular characteristics that examples have in common, they may connect a pattern with a misconception and thereby learn an erroneous procedure.

When multiplication with fractions is introduced, students often have difficulty believing that correct answers make sense; throughout their previous experiences with factors and products, the product was always at least as great as the smaller factor. (Actually, the product is noticeably greater than either factor in most cases.) In the mind of these students, the concept *product* had come to include the idea of a greater number, because this was common throughout most of their experiences with products.

From time to time an erroneous procedure produces a correct answer. When it does, use of the error pattern is reinforced. For example, a student may decide that “rounding whole numbers to the nearest ten” means erasing the units digit and writing a zero. The student is correct about half of the time!

There are many reasons why students tend to learn patterns of error. It most certainly is not the intentional result of our instruction. Yet all too often, individual
students do not have all the prerequisite understandings and skills they need when introduced to new ideas and procedures. When this happens, they may “grab at straws.” A teacher who introduces paper-and-pencil procedures while a particular student still needs to work out problems with concrete aids encourages that student to try to memorize a complex sequence of mechanical acts, thereby prompting the student to adopt simplistic procedures that can be remembered. Because incorrect algorithms do not usually result in correct answers, it would appear that a student receives limited positive reinforcement for continued use of erroneous procedures. But students sometimes hold tenaciously to incorrect procedures, even during instruction that confronts their beliefs directly.

Each incorrect algorithm is an interesting study in itself. In Part Two of this book, you will have opportunities to identify erroneous computational procedures and consider possible reasons students have adopted them. A discussion of research on errors in computation, especially error patterns, is in Appendix B.

Keep in mind the fact that those who learn erroneous patterns are capable of learning. Typically, these students have what we might call a learned disability, not a learning disability. The rules that children construct are derived from their search for meaning; a sensible learning process is involved. This is true even for the erroneous rules they invent, though such rules may involve a distortion or a poor application.

More than procedural learning is in view here. Students often invent similar rules when introduced to the sign for equals. For example, they may decide “The equals sign means ‘the answer turns out to be.’ ”

**Overgeneralizing**

Many misconceptions and erroneous procedures are generated as students overgeneralize during the learning process.

Most of us are prone to overgeneralize on occasion; we “jump to a conclusion” before we have adequate data at hand. Examples of overgeneralizing abound in many areas of mathematics learning. Several interesting examples were observed by project staff at the University of Maryland during their study of misconceptions among secondary school students. At the University of Pittsburgh, Mack studied the development of students’ understanding of fractions during instruction, and she also observed students overgeneralizing.

Consider the following examples of overgeneralizing.

- What is a sum? Sometimes students decide that a sum is the number written on the right side of an equals sign.

\[
4 + 2 = 6 \quad \text{Both are considered sums.}
\]

\[
6 - 2 = 4
\]
• Consider students who believe that all three of these figures are triangles.

Graeber reports an interesting speculation on this situation. These students may be reasoning from a definition of triangle position. Extension of this definition to simple closed curves that are not polygons may lead to this error of including such shapes in the set of triangles.8

• Sometimes students are exposed to right triangles like these . . .

. . . and conclude that a right angle is oriented to the right as well as measuring 90 degrees.

• The student who believes that $2y$ means $20 + y$ may be over-generalizing from expressions like $23 = 20 + 3$. 

• Other students always use ten for regrouping, even when computing with measurements.

\[
\begin{array}{c}
3 \text{ gal. 2 qt.} \\
-1 \text{ gal. 3 qt.} \\
\hline
\end{array}
\quad \rightarrow 
\begin{array}{c}
\frac{3}{4} \text{ gal. 1} \text{ qt.} \\
-1 \text{ gal. 3 qt.} \\
\hline
\end{array}
\]

• Secondary school students sometimes think of the longest side of a triangle as a hypotenuse. They assume the Pythagorean Theorem applies even when the triangle is not a right triangle.

![Diagram of triangle with sides labeled 2, 3, and x, and equation \(x^2 = 2^2 + 3^2\).]

**Overspecializing**

Other misconceptions and erroneous procedures are generated when a student overspecializes during the learning process. The resulting procedures are restricted inappropriately. For example, a student may decide that in order to add or subtract decimals, there must be the same number of digits on either side of the decimal point. Therefore, the student will write 100.36 + 12.57 as 100.36 + 125.70.

Also, students know that in order to add or subtract fractions, the fractions must have like denominators. Sometimes students believe that multiplication and division of fractions require like denominators.

It is quite common for students to restrict their concept of altitude of a triangle to only that which can be contained within the triangle.

![Diagram showing student's response vs. correct response for altitude of a triangle.]

As we diagnose students who are experiencing difficulty, we need to be alert for both overgeneralization and overspecialization. We need to probe deeply as we examine written work—looking for misconceptions and erroneous procedures that form patterns across examples—and try to find out why
specific procedures were learned. Our discoveries will help us provide needed instruction.

**ENCOURAGING SELF-ASSESSMENT**

As teachers, we have an important role in the diagnosis of areas where our students need instruction, but each student also has a significant role in diagnosis. Self-assessment is a powerful tool for teaching and learning, and it may be the most important aspect of the assessment process.

The most effective assessment of all is that of one’s own learning. One of the most valuable lifelong skills students can acquire is the ability to look back and reflect on what they have done and what they still need to do. Students who develop a habit of self-assessment will also develop their potential for continued learning.11

After our students complete an assignment, we may want to hold a debriefing session to help them reflect on what they did. A debriefing can take the form of large or small group discussions, or individual interviews. Questions can be posed that will help our students evaluate their experiences. Although specific questions may help students make judgments about relevant data, we need to include open-ended questions like:

- How do you think you did with this assignment?
- What does someone need to know to be able to do this assignment?
- What was easy for you in this assignment?
- What was difficult for you?

Self-assessment can also be facilitated if each student has a mathematics portfolio. (See “Using Portfolios to Monitor Progress During Instruction” in Chapter 3 for specific suggestions.)

Another way we can help our students acquire the habits of mind needed for self-assessment is to provide experiences with checklists. Students can use checklists as they reflect upon and comment about their own written work. Figure 2.5 is an example of a brief checklist completed after an assignment with paper-and-pencil computation; Figure 2.6 is an example of a longer checklist to be completed by students and turned in with a division assignment when completed.

We can also involve our students in self-assessment by using a questionnaire designed to follow up a particular assignment. Figure 2.7 is an example of a questionnaire for students to use after they have solved a set of nonroutine problems.

Self-assessment is involved when students score written work with the help of a rubric that focuses on more than correct answers. Figure 2.8 is an example of a rubric used with a practice assignment for division by a one-digit number. After students complete the assignment, they score it twice: students determine the number of correct answers first, then they determine a rubric score.
Turn in this checklist with your assignment.

Name ___________

Date ____________

1. My digits are written in place-value columns. Y ? N
2. Others can read my numerals. Y ? N
3. Sometimes I was “stuck.” Y ? N
4. I checked my answers. Y ? N
5. Describe a situation in which this computation could be used.
   ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
   ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

Comments

Less-structured comments like those written in student journals often involve self-assessment. Journals, along with selected student written work, can be filed in individual assessment portfolios. Include in assessment portfolios any checklists, questionnaires, or rubrics students have completed. Later, have students examine these papers and reflect on what they find. Have they grown in their ability to assess themselves?

FIGURE 2.6 Checklist for self-assessment with a division lesson.

Turn this in with your assignment.

Name ___________

Date ____________

1. My digits are written in place-value columns. Y ? N
2. Others will be able to read my numerals and my writing. Y ? N
3. I checked my work to learn which answers are correct. Y ? N
4. Sometimes I don’t know how to start or what to do. Y ? N
5. I think I will be able to really use division. Y ? N
6. Sometimes I give up if the problem is hard. Y ? N
7. I like to do division problems. Y ? N
8. I used multiplication to check my work. Y ? N
9. Dividing is often easy for me. Y ? N
10. I like to work alone on problems like these. Y ? N

Comments
FIGURE 2.7 Questionnaire for self-assessment.

Turn in this questionnaire with your assignment.

Name ___________
Date ____________

1. What mathematics did you use to solve these problems?

2. Did you use drawings or manipulatives to help you solve the problems? If so, describe how you used them.

3. Did you use a calculator? If so, how did you use it?

4. Explain how you solved Problem 3.

5. Did you get “stuck” at any place with Problem 3? If so, tell about it.

6. How do you know your answer to Problem 3 is correct?

7. Are there other correct answers for Problem 3?

Comments

INTERVIEWING

Interviews done sensitively and with demonstrated interest in the student are an effective way to collect information about a student’s mathematical concepts, skills, and dispositions. They are a way to gain both quantitative and qualitative data about an individual.

Consider the following vignette of the first part of an interview. It is October, and Ms. Barnes is interviewing Dexter while other students are working individually and in groups. Dexter was recently assigned to Ms. Barnes’ third-grade class. Dexter and Ms. Barnes are seated at a table in a corner of the classroom; Ms. Barnes faces the center of the classroom and Dexter faces her.
FIGURE 2.8  Rubric for division by a one-digit number.

<table>
<thead>
<tr>
<th>Turn in this rubric with your assignment.</th>
<th>Name ___________</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Date ____________</td>
</tr>
</tbody>
</table>

Look at all of the examples in the assignment. Next, read all five paragraphs. Then decide which paragraph you think best describes what you have done. Finally, circle the number of points in front of that paragraph.

1. I did a few examples, but I did not complete all of them.

2. I did all of the examples. Several do not have correct answers. Digits are not always in place value columns.

3. I did all of the examples. One or two do not have correct answers. Digits are written in place value columns.

4. All examples have correct answers. Digits are written in place value columns.

5. All examples are correct, and I can show why they are correct with base ten blocks. I can also write a story problem for examples like these.

The class is learning to add with regrouping. Dexter computes as follows:

\[ \begin{align*}
\text{A.} & \quad \begin{array}{c}
43 \\
+75 \\
\hline
118
\end{array} \\
\text{B.} & \quad \begin{array}{c}
87 \\
+49 \\
\hline
1216
\end{array}
\end{align*} \]

**MS. BARNES:** I need to find out more about how you add, so we can plan our work together. 
(She shows him his paper and points to Example A.) Tell me, how did you add this example? The sum is correct, but I need to know how you added. I'll write the problem and you can add. Think out loud so I can learn how you added.

**DEXTER:** Three and five is eight. 
(He writes “8” below the line.) Four tens and 7 tens is eleven tens. 
(He writes “11” below the line.)

**MS. BARNES:** And what is the sum? What is the total amount?
**DEXTER:** Eleven tens and 8 ones.
Interviews are worth the time they take because they enable teachers to:

- Gain insights into the student’s understanding of concepts and procedures and identify any misconceptions or error patterns,
- Observe how he or she reasons,
- Learn how well the student can communicate mathematical ideas, and
- Discover the student’s disposition toward mathematics.

As we assess a student’s disposition toward mathematics, we should focus on more than interest in mathematics; we need to look for evidences regarding confidence, curiosity, flexibility, inventiveness, and perseverance.

An interview is not just “oral testing” to determine whether a student can do a task. When we interview a student, we need to think like an assessor and ask ourselves questions like these listed by Wiggins and McTighe:

- What would be sufficient and revealing evidence of understanding?
- How will I be able to distinguish between those who really understand and those who don’t (though they may seem to)?
- What misunderstandings are likely? How will I check for these?¹²
An interview is not a time for expressing our opinions or asking questions prompted by mere curiosity. Rather, it is a time to observe the student carefully and a time to listen. We need to avoid giving clues or asking leading questions. It has been said that we are all born with two ears and one mouth, and we probably should use them in that proportion. This applies quite specifically to us as teachers, because we sometimes want to talk and explain when we should listen.

The pace of the interview should be adapted so the student will respond comfortably.

**Getting at a Student’s Thinking**

Interviews can vary widely in regard to the way we ask a student to respond, but generally we want to encourage students to respond with as much detail as possible. When we ask a student to choose among alternatives that we present, we can ask why the particular choice was made.

If we are to get at a student’s thinking, we need to have the student comment on his or her own thought process. This can be accomplished through either introspection or retrospection.

When eliciting introspection, ask the student to comment on thoughts as the task is being done; have the student “think out loud” while doing the task. For example, ask, “What do you say to yourself as you do this? Say it out loud so I can understand, too.” But when using introspection, the very process of commenting aloud can influence the thinking a student does.

On the other hand, when eliciting retrospection, do not ask the student to comment on thoughts until after the task is completed. Then have the student explain the problem situation in his or her own words: ask how the task was done, and why it was completed the way it was. Try to determine the reasoning used. But when using retrospection, remember that the student may forget wrong turns that were taken.

It is probably best to elicit introspection part of the time and retrospection part of the time.

Following is a transcript of part of an interview I had with a fourth grader. We were discussing what she had written:

\[
\begin{align*}
\frac{1}{3} + \frac{1}{4} &= \frac{4}{12} + \frac{3}{12} = \frac{7}{12}.
\end{align*}
\]

**TEACHER:** What do the equal signs tell us?

**STUDENT:** They tell us . . . that you just do the answer.

**TEACHER:** Which is more, one third or four twelfths?

**STUDENT:** (pause) one third? . . . no . . .
This student knows a procedure, but her understanding of the concept *equals* is inadequate. The interview may even help her evaluate her own thinking.

Many mathematics educators believe that what a student knows about herself as a learner and doer of mathematics—and how she regulates her own thinking and doing while working through problems—can affect her performance significantly.

Some of the questions we ask while interviewing a student will help the student become more aware of her own cognitive processes. For example:

- How did you get your answer? I may have missed something.
- Why is your answer correct?
- If someone said that your answer is not correct, how would you explain that it is correct? Could you explain it another way?
- Can you make a drawing to show that your answer is correct?
- If you had to teach your brother to do this, how would you do it? What would you say to him?

Garofalo lists other questions that can help students become more aware of their cognitive processes. His questions include:

- What kinds of errors do you usually make? Why do you think you make these errors? What can you do about them?
- What do you do when you see an unfamiliar problem? Why?
- What kinds of problems are you best at? Why?
- What kinds of problems are you worst at? Why? What can you do to get better at these?\(^{13}\)

In addition to interview questions, we can use written responses (such as journal entries) to increase awareness of cognitive processes.

By requiring students to examine the processes carefully that they are going through and to verbalize them on paper, the teacher (or other students) can follow the students’ algorithms and find the hidden bugs in their thought processes.\(^{14}\)

Whenever we try to get at a student’s thinking, we should try to focus not only on what the student is thinking but also on what the student understands about his or her own knowledge. Bright comments about this perspective.

I now think about diagnosis as helping learners understand both the power and the limits of their knowledge so that they will know how they need to adjust their knowledge base to fit the problems they have to solve. In this view, learners assume substantial responsibility for the quality of learning . . . teachers set tasks and ask questions that have a high potential to reveal the power and limits of students’ knowledge. Knowing what tasks and questions to pose requires a deep understanding of students’ thinking.\(^{15}\)
When students derive answers to problems, we not only need to get at their thinking in order to understand how they obtained those answers, we also need to learn how they justify their answers—how they prove they are correct in their own thinking. We can look for the three kinds of justification schemes identified by Sowder and Harel\textsuperscript{16} and illustrated by Flores\textsuperscript{17}.

- **Schemes that are externally based**, in which a textbook or authority figure is cited as justification.
- **Schemes that are empirically based**, in which students use perception or concrete objects to show that their answer is correct.
- **Schemes that use analysis**, in which students use counting strategies or cite mathematical relations to justify their answer.

As a student’s thinking develops over time, we expect to see fewer uses of justification schemes that are externally based. We even hope to see use of empirically-based schemes eventually give way to schemes that use analysis, for such thinking is distinctly mathematical.

### Observing Student Behavior

As we interview our students, we observe their behaviors. Here the term *observe* refers not only to seeing, but also to listening attentively. Observing a student is more complex than it sounds, because students often develop defense mechanisms to cover their confusion and to make us believe they understand even when they do not. Observations may be quite informal, interactive, or they may be more structured.

**Informal Observations.** Informal observations take place wherever we have opportunities to observe students. These observations can be a source of information, whether the student is engaged in a classroom lesson, participating in a cooperative group, working at a learning center, or playing on the playground.

Our students bring informal mathematical knowledge to school settings. Note how your students get the mathematical information they need as you watch them play games, plan bulletin boards, or do other activities. Listen to their conversations, and pose a diagnostic question from time to time.

**Interactive Observational Assessment.** Diagnosis and instruction are both involved in this interactive approach to teaching.

Students are engaged in a mathematical problem while the teacher circulates among them to observe and make note of their work. Rather than nod and say, “Yes, you have it” or tell the students that they “have not found the solution yet,” the teacher responds to them by writing questions that challenge their thinking and mathematical reasoning.\textsuperscript{18}
When we use this approach with students in our classes, we observe their behavior as they work and also observe their responses to the written questions we give them to ponder.

**Structured Observations.** Interviews that involve more structured observations take place within contexts as varied as clinical settings and one-on-one interaction between a student and a classroom teacher or aide.

The *structure* in a structured observation comes from a script for the interviewer. The actual words used in the interview may or may not vary from the script, depending upon the purpose of the interview. (Research studies are more apt to require strict adherence to a planned script.) Even when we use a script of some kind, we must remember that we are involved in structured observation. Remember to keep your eyes and ears open as the student responds, and make appropriate records!

Here is an example of a simple script; it was used to diagnose a student who is experiencing difficulty solving verbal problems.

- Read me the problem, please.
- What is the question asking you to do?
- How are you going to find the answer?
- Do the work to get the answer and tell me about your thinking as you work.
- Write down the answer to the question.$^{19}$

**Recording Student Behavior**

A record of responses needs to be made during an interview. This can be done by writing notes and/or by audio or video recording. (Do not rely on your memory to make a record at a later time.) Even when a recording is made, it may be wise to supplement the recording with written notations that describe only those things that will *not* appear on tape. We may also want to record our judgments about the student’s level of understanding.

Written responses can take different forms, but regardless of the form, they should end up in some kind of student assessment folder so we can look for patterns across different kinds of responses.

Notes can be written on three-by-five-inch cards. Cards such as the one shown in Figure 2.9 are useful for making records of brief interviews during instruction. Or we can keep an observation sheet for each student, similar to Figure 2.10. An advantage of an observation sheet is that it gives us a single record over time, a record we can easily share with parents. A disadvantage is that a particular student’s sheet is not likely to be readily available when we want it; we may have to make a quick note and later transfer it to the observation sheet.

When conducting a more structured interview, we may want to write notes in a space provided at the side of our planned question. Our script can be on a sheet of paper or on a set of cards. Figure 2.11 shows what one card might look like.
FIGURE 2.9  A simple observation form for general use.

<table>
<thead>
<tr>
<th>Name ___________________________</th>
<th>Date ______________</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observation/Interview</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Activity</th>
<th>Observed Behavior</th>
<th>Suggestion for Instruction</th>
</tr>
</thead>
</table>

FIGURE 2.10  An observation sheet for an individual.

<table>
<thead>
<tr>
<th>Date</th>
<th>Activity</th>
<th>Observed Behavior</th>
<th>Program Suggestions</th>
</tr>
</thead>
</table>


We need to plan ahead so we will be able to make records of pertinent observations. Forms such as the examples in Figures 2.9–2.11 can be adapted to fit our own situations.

Watching Language: Ours and Theirs

Usually, written and oral language in mathematics is grammatically simple, but sometimes it is more complex linguistically. Even with young children, an expression as simple as $2 + 3$ is interpreted in varied ways.

- two and three
- two plus three
- two and three more
- three more than two
As adults we know that these expressions are equivalent, but young students are confused by such a diversity of interpretations.

What we say is sometimes complex linguistically because we tend to use pre- and post-modifiers. Needlessly complex expressions abound and include sentences like:

- Find the pair of numbers whose product is greater than 100.
- The value of this digit’s place is one tenth of the value of what place?

It is also true that we verbally interpret in different ways the question asked by an open number sentence. For example, \( N - 28 = 52 \) might be expressed as “28 less than what number is 52?” or as “What number less 28 is 52?” Sometimes our students do not know what we mean because of the way we say it.

How might a particular student say it? We may be able to gain an insight into his or her use of language by having the student read or interpret an expression with the same structure, but with single-digit numbers. For example, before asking the question posed by \( N - 28 = 52 \), show \( N - 3 = 5 \) and ask, “What question does this ask?”

The way we use language is crucial not only as we assess, but whenever we teach.

Mathematics education begins and proceeds in language. It advances and stumbles because of language, and its outcomes are often assessed in language.\(^{20}\)

As we teach second language students, we need to remember that they may tend to think in different categories and make different associations.

The way we use language is also crucial as we teach people with disabilities. What we say may enhance their dignity or it may reflect stereotypes. Furthermore, language is constantly changing, and we need to be sensitive to current usage.\(^{21}\)
Probing for Key Understandings

When we need to learn what a particular student understands about specific ideas, we can solicit evidence of understanding by asking a question or presenting a carefully designed task for the student to do; then observe the student's response.

Likely, we will need to present follow-up questions or related tasks before we can make appropriate inferences regarding what the student understands. Think how the particulars in the task could be changed a bit to create a related task.

Rather than always focusing on computation skills per se, we frequently need to focus on concepts related to number sense; for example, on understanding the operations of arithmetic—concepts that enable students to know which operation to use when solving problems.

During such interviews, how might we get at a student's understanding of each of the following stated key ideas? What evidence of understanding might we elicit? What could we say or do? Possibilities are illustrated for each statement: a question that could be asked or a task that could be presented follows the statement. (If you try these with your students, you may have to adapt them to the appropriate level.)

- A digit's value in a numeral is determined by the place where it is written.

With base ten blocks at hand, show the numeral “243” and say: “Can you show this much with the blocks? Try it.” Then, “How do you know which blocks to use for the two?

- The values of places within a numeral are powers of ten in sequence.

Show a numeral for a whole number (as great as appropriate) and say, “Start at this end, and tell me the values of each place within the numeral.” Then, “Can you do it if you start at the other end? Try it.” Then, “Is there some kind of pattern? Can you tell me about it?”

- The value of a numeral for a whole number is the sum of all the products (face value × place value) for each digit.

Lay out a collection of base ten blocks and show the numeral “2453” saying, “Show me this much with the base ten blocks.” Then repeat the procedure with a place value chart or an abacus. Next say, “How do you know how much the numeral shows?” After the student responds, you may want to ask, “As you decide what the number is, do you add? . . . or subtract? . . . or multiply? . . . or divide? Think about it.”
• Equals means “the same as.”

Show the equation \(25 + 12 = \Box\) and ask, “Can you tell me the sum?” Then point to the equals sign and ask, “What does this mean?” (If the student responds “equals,” ask, “What does that mean?”) Then present the equation \(\Box = 13 + 22\) and ask, “Can you tell me the sum?” Again, point to the equals sign and ask, “What does this mean?”

The student who thinks equals means “results in” rather than “is the same as” may respond to the second equation: “You can’t do that.”

• Addition tells the sum if you know both addends.

Show the equation \(3,278 = 5,190\) and ask, “How would you find the missing number? Then ask, “Why would you do that?”

• Division tells the missing factor if you know the product and only one factor.

Show the equation \(\Box \times 624 = 1,872\) and ask, “How would you find the missing number?” Then ask, “Why would you do that?”

• A fraction in which the numerator and denominator name the same number is a name for one.

Show a number line for whole numbers 0–100, then also show the numeral \(\frac{3}{3}\). Point to the fraction and say, “Can you point to where this amount is on the number line?” Then, “Can you write other fractions for one?”

### Designing Questions and Tasks

When we present a question or a task to a student during an interview, we actually provide a stimulus situation to which the student responds. Stimulus situations can be presented in varied modes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Verbal</td>
<td>Words, oral and written</td>
</tr>
<tr>
<td>Written symbols</td>
<td>Numerals</td>
</tr>
<tr>
<td>Two-dimensional</td>
<td>Paper-and-pencil diagrams, photographs</td>
</tr>
<tr>
<td>representations</td>
<td></td>
</tr>
<tr>
<td>Three-dimensional</td>
<td>Base ten blocks, math balance, place value</td>
</tr>
<tr>
<td>representations</td>
<td>chart</td>
</tr>
</tbody>
</table>
Each stimulus calls for a response; responses for a given stimulus can be similarly varied among modes. We increase our confidence in what we learn about a student when we use a variety of questions and tasks to elicit evidences of understanding.

Make sure directions clearly indicate what is expected. Tasks should engage students and elicit their best performances.

Following are things we can do that may help us obtain information we will get no other way:

1. Say, “This time I’ll hold the pencil and you tell me what to do.”
2. Have students describe to other students how they solved a problem, or have them write their descriptions on paper.
3. Provide a slightly different context, and ask students to use the idea.
4. Sometimes we can have a student explain a graphic organizer she has made: a number line, a cognitive map, a flow chart, etc.
5. We may want to ask the student to tell how he would explain the idea or procedure to a younger sibling, or have him make a poster that explains what he did.
6. At different times, it may be helpful to say, “Your answer is different from mine. I could be wrong and you could be correct. Show me that yours is correct.” Or “Can you show me another way?”
7. We can often get useful information about what a student understands by asking questions like:
   “How do you know that $4/9 \times 60$ is $< 30$?”
   “How do you know that $1/4 > 1/5$?”

An interview typically involves rather focused questions and tasks designed to gain information regarding specific concepts and skills the student may or may not possess; but we can often gain a more complete understanding of a student’s mathematical thinking by including items that are more open-ended. Examples like these are sometimes used as warm-ups or as homework.

• Instead of presenting $4 \times 17 = ?$ ask the student to find two numbers with 108 as their product: a one-digit number and a two-digit number.
• Instead of presenting diagrams and asking which diagrams represent $3/4$, ask the student to draw several diagrams or pictures that show $3/4$.

Furthermore, as we design our questions and tasks, we need to remember:

• We may need to diagnose a student’s ability to estimate; instruction may need to focus on this important skill and the concepts involved.
• Realistic contexts in assessment as well as instruction can help to engage and motivate students.22
• Students with learning disabilities often communicate information that is incorrect, yet it is what they actually see.
A procedure for planning, conducting, and reporting a diagnostic interview is described in Appendix E. A rubric is provided for instructors who use it as an assignment.

When you are conducting a diagnostic interview:

**Be sure to**

- Find a place with few distractions or interruptions, and have the student face you with his or her back toward potential distractions.
- Start with easier tasks, and very gradually present more difficult material.
- Accept the student’s responses without judging. Use neutral feedback such as a nod of the head.
- Ask probing questions in response to both correct and incorrect answers. Keep your tone of voice the same, whether the response is correct or not.
- Encourage the student to “think out loud” to solve problems in different ways and to verify answers.

**But do not**

- Tell the student that errors have been made.
- Interrupt the student, or begin to teach what is “correct.”
- Give praise as feedback. You may want to say, “Thank you” or “That was helpful.”

**Using Graphic Organizers for Diagnosis**

Diagnoses of individuals regarding understandings and skills should be based on performance data. Checklists, questionnaires, and journal entries used for self-assessment; written assignments; project results; interviews; and other items in an assessment portfolio all provide useful data for making such diagnoses.

So that students with different intelligences and learning styles can demonstrate what they understand and are able to do, our assessment tasks should be varied in format. A graphic organizer often provides a useful format for an assessment task; it can focus on relationships while requiring fewer verbal skills.

Graphic organizers that have been used during instruction are especially useful for diagnosis. When numeration has been related to number lines, for example, a number line task can be used to help determine what the student understands about numeration (see Figure 2.12). Figure 2.13 is another assessment item focusing on numeration concepts, but it is based on a cognitive map.
Write the missing numerals in the boxes on each number line.

95 100 [ ] 110 115

899 [ ] [ ] 902 903

995 [ ] [ ] 1010 1015

FIGURE 2.12  A number line used for assessing numeration concepts.

If our students have had experiences interpreting flow charts, we can construct performance items from flow charts they used or from similar flow charts. Figures 2.14 and 2.15 are examples of items based on flow charts.

We may also want to have students make concept maps to communicate what they know about mathematics. A concept is written on paper, then relationships are
shown with lines. Linking words (usually verbs) also can be added. It is important that students put their thoughts on paper. Figure 2.16 points to a fourth-grade boy's very limited understanding of “subtraction.” And Figure 2.17 shows how another fourth grader responded to “fractions.” She associated fractions with drawings, which she labeled incorrectly.

**USING TESTS**

Although effective assessment is integrated with teaching day by day, there are times when a more focused and complete look at student performance is helpful. For example, we may administer a diagnostic test when we begin working with a new class of students, when a new student is assigned to us, or when a student is experiencing difficulty.

Any diagnostic test we use should be curriculum-based. It is within the specific areas of mathematics that constitute our curriculum that we need to learn
Here is part of a flowchart we made in class. Write the missing question in the diamond shape.

**Adding Two Fractions**

Start

Add the numerators.

- yes
- no

Write one fraction above the other.

**FIGURE 2.15** A teacher-made item focusing on procedural knowledge for adding fractions.

**FIGURE 2.16** Cognitive map for subtraction by a fourth-grade boy.
about a student’s strengths. We may then plan needed instruction that will build on those strengths.

As we diagnose what students understand about mathematics and their skills in problem solving and computation, part of what we learn comes from their written work. Even when calculators are used in assessment, arithmetic computation is often tested on a non-calculator portion of an assessment procedure. Student papers shown in later chapters in this book illustrate some of what we can learn by examining paper-and-pencil assignments very carefully.

When appropriate, diagnostic data gathering may include a test—a sequence of performance tasks—to be completed by individuals. Whether administered individually or to a group of students, a diagnostic test is one of the many forms of data gathering that can help us learn about individual strengths within selected areas of mathematics.

Diagnostic testing is formative assessment; it is designed to refocus instruction on important concepts and skills needed by individual students. But administering a test is not enough: “... tests alone cannot create improvement.” 23 We need to actually make adjustments for individual learners—changes in curriculum and instruction implied by the data we collect.

State accountability tests usually provide very limited diagnostic feedback regarding the instructional needs of individuals. Popham notes:

Most state accountability tests fail to produce the kinds of data that will improve teaching and learning. Teachers can get the data they need from classroom assessments—if they know how to design instructionally useful tests. 24

Standardized achievement tests have limited value for diagnostic purposes. They can help identify broad areas of strength and weakness, thereby serving as a springboard to further assessment. For example, they might show a student
performing at grade level in one operation but not in another. However, they usually sample such a broad range of content that we are not likely to learn what we need to know about specific concept and skill categories. According to Kamii and Lewis, achievement tests emphasize lower-order thinking and can result in misleading information—at least in the lower grades.\textsuperscript{25}

Commercial diagnostic tests are available: examples include Early Math Diagnostic Assessment (EMDA)\textsuperscript{26} for Pre-K through Grade 3 students; and \textit{KeyMath-R/N: (A Diagnostic Inventory of Essential Mathematics—Normative Update)},\textsuperscript{27} which has an attractive format and is intended for students ages 5–22 years. Such commercial diagnostic tests are designed to help us identify and plan appropriate instruction for individual students.

Computers are powerful tools, and many teachers hope computers will be able to help them diagnose the strengths of their students, thereby assisting them as they plan instruction. But software has not greatly advanced beyond a set of paper-and-pencil tests. We need to make sure that the computer programs we use are more satisfactory for our purposes than paper-and-pencil tests—including brief, focused tests we could design ourselves.

It is possible to enter a student's written computational work into a computer and then analyze it; but when this is done, other variables, such as keyboarding ability, are introduced.

\textit{STAR Math}\textsuperscript{28} is a recent attempt to provide a computer-aided math assessment program that we can use with our students. Branching technology is used so that each student can be tested as quickly as possible. However, the tests focus more on appropriate levels of instruction for each student and preparing useful reports than on diagnosis of very specific knowledge and skills.

As we interpret diagnostic test performances, especially in the area of computation, we must try to distinguish between a student's lack of conceptual understanding and his not knowing a correct procedure. For example, among those students who are learning to add with renaming, one student may understand that a two-digit number consists of tens and ones, but he records the two-digit sum at the bottom in the ones column. That student needs to be taught that only one digit can be written in each place. On the other hand, another student may write “1” at the top of the tens column (correct procedure) but not actually understand that the sum for the ones column is so many tens and ones.

Most often, we do not have time to interview each student to determine their knowledge and skill related to a particular concept or procedure; so, from time to time, we probably need to prepare our own assessment items for a specific concept or skill category. For instance, we may need a short test to administer to those students who have experienced some difficulty subtracting when regrouping is involved.

The error patterns presented in Chapter 4 suggest distractors we can use when constructing diagnostic test items for subtraction of whole numbers. Distractors drawn from common error patterns may give us clues to students' thought processes. For example, the following multiple-choice item was built from error patterns: each distractor is an answer a student might choose if she has learned an
erroneous procedure. When you have an opportunity, ask individual students why they chose their answers.

The answer is:

4372
–2858
  a. 2526
  b. 1514
  c. 524
  d. 2524

**GUIDING DIAGNOSIS IN COMPUTATION**

A student’s work not only must be scored, it must be analyzed if it is to provide useful information. Whenever someone else marks examples correct or incorrect (students or an aide), we can spend more of our time analyzing student work and planning needed instruction.

Observe what a particular student does and also what the student does not do; note computation with a correct answer and also computation that has an incorrect answer; and look for those procedures that might be called mature and those that appear less mature. Distinguish between situations in which the student uses an incorrect procedure and situations in which he does not know how to proceed at all.

Following are principles to keep in mind as we diagnose the work of students who are having difficulty with computation.

1. **Be accepting.** Diagnosis is a highly personal process. Before a student will cooperate with us in a manner that may lead to lessening of problems with computation, he must perceive that we are interested in and respect him as a person, that we are genuinely interested in helping him, and that we are quite willing to accept a response—even when that response is not correct. We must exhibit something of an attitude of a good physician toward his patient. As Tournier, a Swiss physician and author noted many years ago, “What antagonizes a patient is not the truth, but the tone of scorn, pity, criticism, or reproof which so often colors the statement of the truth by those around him.”

2. **Focus on collecting data.** It is true that assessment is a continuous process; even during instruction we need to keep alert for evidence that a student does not understand or has learned an error pattern. Even so, there are times when we need to make a focused diagnosis, and at those times, we must differentiate between the role of collecting data and the role of teaching—we need to collect data, but not instruct. Diagnosing involves gathering as much useful data as possible and making judgments on the basis of data collected; in general, the more data, the more adequate the judgments which follow. A student is apt to provide many samples of incorrect and immature procedures if he sees that we are merely collecting information that will be used to help him overcome his difficulties. However, if we point out errors, label responses as “wrong,” and offer instruction while collecting data, he is far less likely
to expose his own inadequate performance. Many teachers tend to offer help as soon as they see incorrect or immature performance. When those teachers begin to distinguish between collecting data and instruction, they are often delighted with the way students begin to open up and lay bare their thinking.

3. **Be thorough.** A single diagnosis is rarely thorough enough to provide direction for ongoing instruction. If we are alert during instruction following a diagnosis, we may pick up cues that suggest additional diagnostic activities. Keep in mind the fact that we need to use varied types of assessment tasks if we are to observe each individual at his or her best.

4. **Examine specific understandings and skills.** More formal assessments, published tests or computer-generated tests may help to identify broad areas of strength and weakness. Let them serve as springboards for further assessment in which more specific concepts and skill performances are examined—often through an interview.

5. **Look for patterns.** Data should be evaluated in terms of patterns, not isolated events. A decision about corrective instruction can hardly be based upon collected bits of unrelated information. As we look for patterns, we look for elements common to several examples of a student’s work—a kind of problem-solving activity. We try to find repeated applications of erroneous definitions and consistent use of incorrect or immature procedures. The importance of looking for patterns can hardly be overstressed. Many erroneous procedures are practiced by students, while teachers and parents assume they are merely careless or “don’t know their facts.”

6. **Discuss progress with parents.** Be sure to help parents understand the full scope of the mathematics curriculum and what their child is learning and will be learning. In regard to areas of difficulty, be sure to discuss the student’s progress. Our conversations with parents often give us additional clues that help us plan instruction.

**Reflecting on Diagnosis of Misconceptions and Error Patterns**

As we teach mathematics, we need to be continually gathering and using information about student learning. This is no less true when we teach concepts and skills related to computation. As we examine students’ papers diagnostically, we look for patterns, hypothesize possible causes, and verify our ideas.

Checklists, rubrics, and questionnaires can sometimes facilitate self-assessment, which is to be encouraged. Published tests may help identify areas of strength and weakness, but interviews are likely to be needed from time to time to get at students’ thinking and probe for specific understandings and skills. We may have to design performance tasks, possibly using manipulatives or graphic organizers, to examine specific concepts.
serious students of diagnosis will want to examine many of the references listed at the end of this book. Diagnosis of misconceptions and error patterns in computation is a continuing process; it interacts with instruction in computation—which is the focus of the next chapter.

REFERENCES

9. Ibid., pp. 4–12.
10. Ibid., pp. 4–30 through 4–35.
26. Harcourt Assessment (see http://www.harcourtassessment.com)
27. American Guidance Service (see http://www.agsnet.com)
28. Renaissance Learning (see http://www.renlearn.com/math)
We have learned that when Fred multiplies whole numbers, he usually adds the “crutch” before multiplying. Now we need to plan instruction that will help Fred.

This chapter is designed to help us provide effective instruction in computation—instruction that is based on the data we collect. We are urged to make sure our students understand numerals before we teach them to compute. Different methods of computation are stressed, as is the importance of using manipulatives appropriately. Varied instructional activities are encouraged: talking and writing math, creating graphic organizers, using calculators and alternative algorithms, and involving peers. Guidelines for instruction are included; and later, in Chapters 4 through 12, specific suggestions are listed for particular error patterns.

The previous chapter focused on diagnosis because it is important to collect varied forms of data and make thoughtful inferences about student learning. But diagnosis must serve instruction. We need “diagnostic teaching” in which diagnosis is continuous throughout instruction. As Rowan asserts, “Diagnosing instructional needs is an integral part of the instructional process...” We can interweave instruction and diagnosis as we teach computational procedures—always alert to what each student is actually doing and eager to probe deeper. We can be willing to change our plans as soon as what we see or hear suggests that an alternative would be more fruitful in the long run. Diagnostic teaching is, first of all, an attitude of caring very much about each student’s learning.
Diagnostic teaching is also cyclical. After an initial diagnosis, we plan and conduct a lesson, but what we see and hear during the lesson prompts us to modify our previous judgments and seek more information before planning the next lesson. Sometimes we move through a cycle very rapidly several times in the course of a single lesson. At other times, one cycle occurs over a span of several lessons.

The instruction we plan should focus on our students’ mathematical thinking—including their thinking about procedures and algorithms. Do students know when and why a particular procedure is used, or if an algorithm always provides correct answers, or why it does?

Our students will learn various computational procedures and the concepts and principles that underlie the different forms of computation; as they learn, they will observe patterns and construct knowledge—but more is needed. If our students are to understand and actually use what they are learning, they must reflect on what they observe and connect it with other mathematical ideas they already know. We must help students not only learn concepts, principles, and procedures, but also help students understand how these are related.

As we teach, we ourselves inevitably model a disposition toward mathematics and learning mathematics. We need to demonstrate an approach to mathematical situations and to learning mathematics that is confident, flexible, curious, and inventive.

**Developing Number Sense**

When students develop a good foundation, including required number concepts and principles, they are ready to learn about operations and computation. What we sometimes call *number sense* is the most basic component of that foundation.

During the early years teachers must help students strengthen their sense of number, moving from the initial development of basic counting techniques to more sophisticated understandings of the size of numbers, number relationships, patterns, operations, and place value.4

The following NCTM expectations for pre-K through second grade suggest what is meant by number sense:

- count with understanding and recognize “how many” in sets of objects;
- use multiple models to develop initial understandings of place value and the base-ten number system;
- develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections;
- develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers;
- connect number words and numerals to the quantities they represent, using various physical models and representations;
- understand and represent commonly used fractions, such as $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$.5
These expectations suggest activities for developing number sense in the early grades, instruction that can provide a good foundation for teaching computation. Much of this instruction can take the form of problem solving.

Even in the early grades we must be alert to what each student already knows and can do—then select activities that are appropriate for our particular students. The instruction we plan needs to be data-driven.

**HELPING STUDENTS UNDERSTAND BIG IDEAS**

Some students have difficulty learning to compute because they do not adequately understand the concepts and principles that underlie algorithms. Their understanding of multi-digit numerals and what the operations mean does not provide the foundation needed to learn procedures that make sense to them. Similarly, when they are introduced to algorithms with fractions, their understanding of fractions and what the operations mean is not adequate for them to make sense of those procedures. Very often, computation procedures that make no sense to a student are not remembered accurately—nor are they used appropriately.

Math resource teacher Mazie Jenkins wrote the following as she thought about students she had interviewed.

I recently interviewed ten fourth and fifth graders whose mathematics instruction had not focused on big ideas. These children all had been taught procedures that didn’t make sense to them, so they didn’t remember them. They often pieced together different algorithms in senseless ways. It was painful to watch them solve problems; I could have cried. . . When they were done, many students looked at me and asked, “Did I do it right?” Children who don’t have an understanding have to look to the outside for validation. They didn’t look inside and think, “I know I did it right.” Unless someone teaches them how to learn mathematics with understanding, they will be lost.  

Learning specific algorithms involves procedural learning, and Chapter 1 emphasized that procedural learning should be tied to conceptual learning. During developmental instruction, we need to encourage our students to think about why they are doing what they are doing as they compute; we need “. . . to require reasoning that justifies procedures rather than statements of the procedures themselves.” Students need to *reflect* on what they are doing. They need to connect what they are learning to what they already know.

In our students’ minds, symbols need to be connected not only to words, but also to concepts and to principles. Equality is an important idea, and we tend to assume our students understand more than they do when they say “equals.” The systems we use for creating numerals for whole and rational numbers involve many concepts, some of which are not easy for young children to understand.
Numeration

Place value is the key to teaching computation with our base-ten numerals, but understanding Hindu-Arabic numerals for whole numbers is not just identifying place values. The concept of values assigned to places is important, but it is only part of what students need to know if they are to understand multi-digit numerals and learn computational procedures readily.

If instruction in numeration is to be data-driven, we must learn what our students already understand about numerals for whole numbers. One thing we can do is write true/false statements such as the following, and ask our students which number sentences are true and which are false. We also need to ask them how they know it is true or false.8

a. 56 = 50 + 6
b. 87 = 7 + 80
c. 93 = 9 + 30

Consider this principle: “A multi-digit numeral names a number which is the sum of the products of each digit’s face value and place value.”9 [For example: \(398 = (3 \times 100) + (9 \times 10) + (8 \times 1)\).] The terms used in this statement alert us to different ideas that are incorporated within multi-digit numerals. To understand multi-digit numerals a student must first have some understanding of the operations of addition and multiplication, and be able to distinguish between a digit and the complete numeral.

Understanding a digit’s face value involves the cardinality of the numbers zero through nine. Place value itself involves assignment of a value to each position within a multi-digit numeral; that is, each place within the numeral is assigned a power of ten. We therefore identify and name the tens place and the thousands place. This, rather specific association of value with place, is independent of whatever digit may happen to occupy the position within a given numeral.

Occasionally, students associate the ones place with the left position within a given numeral. Their teacher may have referred to the ones as the “first” position. If she did she was probably thinking “on the right,” but her students assumed it was “on the left” because they normally proceed left to right—as when they read. We must make sure that our explanations are clear, and solicit feedback (formative assessment) to make sure our students understand correctly.

Sometimes students having difficulty with whole number computation can identify and name place values, but they cannot get the next step. They have not learned to combine the concepts of face value and place value. It is the product of a digit’s face value and its place value, sometimes called “total value of the digit” or “product value,” which must be used. The sum of such products is the value of the numeral. In renaming a number (as we often do when computing) these products of face value and place value must be considered continually; and while considering these things, our students also need to think about the numeral as a whole.
Children do not quickly develop the conceptual structures associated with our place-value system for writing numbers; it takes a long time. Jones and others identified four key constructs that “appear to be central to the development of multi-digit number sense—counting, grouping, partitioning, and ordering numbers.”¹⁰

When we teach our students about numerals, we should introduce numerals as a written record of observations made while looking at or manipulating objects. For multi-digit numerals for whole numbers, these observations frequently follow manipulation of materials according to accepted rules in order to obtain the fewest pieces of wood (or the like). We may need to trade ten objects for one object that is equivalent to the ten if we can, or we may be required to exchange chips in a trading game. In this way, representations for the standard or simplest numerical name for a number are obtained.

When students associate a numeral with concrete aids, it is important that they have opportunities to “go both ways.” On the one hand, students may be given materials to sort, regroup, trade, and so on, and then record the numeral that shows how much is observed. But they also need to be given a multi-digit numeral to interpret by selecting or constructing materials that show how much the numeral represents. If our students are able to go from objects to symbol and also from symbol to objects, they are coming to understand what multi-digit numerals mean.

Initially, we should have our students work with concrete aids that make it possible to compare the value of a collection of objects with the equivalent value of a single object (e.g., bundled sticks, or base blocks). Later, they can use aids in which many objects are traded for a single object—an object that is identical except for its placement (e.g., sticks in place-value cans, or trading activities with chips of one color on a place-value mat). These aids are helpful because they more accurately picture the way digits are used within multi-digit numerals.

In our base-ten numeration system, the value represented by each digit involves a relationship with the unit. This is true not only for numerals for whole numbers that state the number of units, but also for decimals which must be interpreted as part of a unit. We must help our students focus on the unit.

Decimals are numerals for rational numbers—but so are fractions and percents. If our students are to use all of these numerals effectively in computation, they need much experience with the varied meanings associated with the numerals—meanings as varied as part of a whole and indicated division. Furthermore, they
need to be able to relate the different kinds of numerals for rational numbers—decimals, fractions, and percents.

**Equals and Equivalent**

When we say *equals*, we mean *the same*, whether we are talking about numbers or not. When two numerical expressions refer to the same number, we say the expressions are *equivalent*. We can express that relationship with the word (or symbol) *equals* because both sides of the equation name the same number.

We say $20 + 4 = 24$ and $8 + 7 = 9 + 6$ and $21 = 15 + 6$. Both $20 + 4$ and $24$ are names for the number we call *twenty four*. There is only one such number (it is one point on the number line) but it can be named many different ways—with different numeration systems, and with mathematical expressions involving various operations. Both symbolic expressions name *the same* number; they are equivalent. Students are taught to say “equals means *is the same as*;” but often it is a rote response that is not applied.

The basic relational concept we call “equals” is difficult for many young students. Early instruction too often encourages students to conceive of equals as a step in a procedure. To them, it actually means *results in*; therefore, $2 + 4 = 6$ becomes “two and (plus) four results in six.” Or it is understood to follow a question where it means, “do it now,” with the answer given next. These students do not think of equals as naming or describing a relationship; instead, they think of equals as an operator indicating a calculation to be done. It is not surprising that the author finds that when presented with an equation like $= 7 + 8$, many young students respond, “You can’t do that.” Or given an equation like $3 + 2 = 4 + 1$ they say, “You can only have one number after equals.”

... many children see only examples of number sentences with an operation to the left of the equal sign and the answer on the right, and they overgeneralize from those limited examples. ... limiting children’s exposure to a narrow range of number sentences does appear to contribute to students’ misconceptions about the equal sign.\(^{11}\)

If they are to enjoy success with arithmetic and with all of mathematics, it is extremely important that students come to an accurate understanding of equality. Many mathematics educators view the understanding of equality as a foundation for algebra.\(^{12}\)

Carpenter, Franke, and Levi suggest benchmarks for us to work toward as understanding of equals evolves.

- Children describe what they think the equals sign means.
- They accept as true a number sentence not in the form $a + b = c$. (for example: $6 = 2 + 4$, or $5 = 5$)
- Children carry out calculations on both sides of the equals sign and compare; that is, they recognize a relationship.
- They compare without carrying out the calculations.\(^{13}\)
Other Concepts and Principles

Other concepts and principles are incorporated within various computational procedures. Our students need to investigate them while studying algorithms, though they do not need to be able to express them with precise language in order to compute. Initially, they can be described informally, but eventually these powerful principles—these big ideas—will need to be expressed using words and symbols.

For example, our students can reflect on the following compensation principles.

- When adding two numbers, if the same number is added to one number and subtracted from the other number, the sum of the two numbers stays the same. \(398 + 552 = 400 + 550 = 950\)
- When subtracting one number from another number, if the same number is added to both numbers (or subtracted from both), the difference remains the same. \(552 - 398 = 554 - 400 = 154\)

Students can investigate other concepts and principles, too, and they can apply them to computational procedures. Some of these principles are properties of operations on numbers.

- When we add, we can reverse the order of the addends without changing the sum. \(87 + 46 = 46 + 87\)
- When we multiply, we can reverse the order of the factors without changing the product. \(38 \times 6 = 6 \times 38\)
- When we add or subtract zero, the result is the number we started with. \(1,000,000 - 0 = 1,000,000\)
- A number minus that same number equals zero. \(367 - 367 = 0\)
- We can multiply in parts. We can distribute multiplication over addition

\[
4 \times 65 = 4 \times (60 + 5) = (4 \times 60) + (4 \times 5) = 240 + 20 = 260
\]

and we can distribute multiplication over subtraction.

\[
4 \times 58 = 4 \times (60 - 2) = (4 \times 60) - (4 \times 2) = 240 - 8 = 232
\]

Clearly, many of these concepts and principles are involved not only in paper-and-pencil computations, but they can help our students estimate and compute mentally. They can also be applied when calculations are simplified.

Using Models and Concrete Materials

The experiences we plan for our students help form their dispositions toward mathematics. Our use of concrete materials, frequently called manipulatives, can contribute to a positive disposition—if those experiences include exploration, problem solving, accurate modeling of the mathematics involved, and reflection by each student.

If a teacher believes that getting correct answers is all that is really important, students will believe that it is all right to push digits around whether it makes any sense or not; they also will push manipulatives around without thinking and
relating them to a meaningful recording procedure. From his analysis of the use of concrete materials in elementary mathematics; Thompson stresses that students “must first be committed to making sense of their activities and be committed to expressing their sense in meaningful ways.”14 Our classroom talk must focus on thinking, even while students use concrete materials; we must create the expectation throughout all of our mathematics instruction that we want to make sense of the procedures we use and whatever we write down.

In general, well-chosen manipulatives can provide a natural working environment for our students as they learn concepts and procedures. This is true for whole numbers, which are very much a part of each student’s environment. With fractions, students have less direct experience and they are more apt to rely on rote procedures, so we need to take special care when selecting materials. Materials that we can use to create more natural environments include chip-trading activities for whole numbers and fraction bars for fractions. Extensive modeling with materials like these during the early phases of instruction usually helps students develop understandings they can apply flexibly.

Increasingly, students are urged to use reasoning and evidence to verify results as they solve problems and compute, rather than simply relying on the teacher to verify answers. Though students can sometimes use mathematical reasoning to provide that evidence, often they are able to provide needed evidence through drawings or manipulatives—if they are encouraged to do so.

We must make sure that whatever manipulatives we select or design are accurate mathematically. Fraction representations, for example, are sometimes inaccurate; although it is often wise to let students construct the models used, we need to make sure fractional parts are equal in area. Base-ten blocks are sometimes mysterious to students, especially the thousands block because they see only six hundreds on the sides. Although base-ten blocks are accurate mathematically, we need to make sure students understand the mathematics accurately.

Students look for commonalties among their contacts with an idea or a procedure, and as they come to understand they pull out common characteristics among their experiences and form an abstraction. Therefore, our students need experiences in which all perceptual stimuli are varied except those that are essential to the mathematical concept or procedure. A cardboard place-value chart may be of great value, but it should not be the only concrete aid we use for numeration activities; other models can be used also, possibly devices made with juice cans or wooden boxes.

We should help our students learn general concepts and procedures rather than ideas or processes specific to a particular model or example. For instance, students need to learn that “ten ones is the same amount as one ten,” rather than “take ten yellows to the bank then put one blue in the next place.” Similarly, if we do not focus on the general procedure, in the specific subtraction number sentence 42 − 17 = 25 a student may conclude that the five units in the answer is simply the result of finding the difference between the two and the seven. Models and examples should be varied so irrelevant characteristics are not observed as common attributes.

As our students use manipulatives to model concepts, we should involve our students in experiences which “go both ways” whenever this is possible. This is
especially important for numeration concepts. Have students manipulate models and record what they observe with symbols, but also let them begin with symbols and interpret the symbols by modeling the concept. For example, provide a collection of base-ten blocks, and point to a unit block and explain, “This block is one;” then give a student a numeral card such as 1,324 and have the student show that amount with the blocks. In contrast, assemble a collection of base-ten blocks: five hundreds blocks, two unit blocks, one thousands block, and three tens blocks. Ask the student to write a numeral for the amount shown with the blocks.

Sometimes manipulatives can be arranged in relation to one another, just as digits are placed in relation to one another when computing on paper. This is true for some of the game-like activities described in Appendix C, and for the way sticks (singles and bundles), base-ten blocks, and place-value charts are often used.

**Helping Students Understand Operations**

Knowing different methods of computation will be of little value to students if they do not understand what each operation does. As our students solve problems in the world around them, they must understand the meaning of each operation if they are to know which operation to compute and which numbers to use within a problem situation.

In *Principles and Standards for School Mathematics*, NCTM emphasizes the need for students to understand what the operations mean and how they relate to each other. During the early grades, students encounter subtraction interpreted as “take away” and as “comparison;” they also encounter what are called “missing addend” situations in which the problem situation may be recorded with a plus sign but subtraction is used to solve the problem (e.g., $24 + \square = 53$). They may also encounter what might be called “missing sum” situations in which the problem situation is recorded with a minus sign but addition is used to solve the problem (e.g., $\square - 37 = 28$). Later, students encounter comparable situations involving multiplication and division.

Meanings for the different operations are often described in terms of structures characteristic of problem situations for particular operations; and these structures can be investigated. While studying addition and subtraction situations, students can explore relationships between the numbers for parts and the total amount. Later, when they study multiplication and division situations, they can investigate relationships between the numbers that tell about equivalent parts and the total amount. The structures they learn for each operation are useful, whether problem situations involve whole numbers or rational numbers. One way of summarizing these structures follows:

- **Addition** tells the total amount (sum) whenever you know the amounts for the two parts (addends).
- **Subtraction** tells the amount in one part (addend) whenever you know the total amount (sum) and the amount in the other part (addend).
• *Multiplication* tells the total amount (product) whenever you know the amount for both numbers that tell about equivalent parts (factors).

• *Division* tells the amount for one number about equivalent parts (factor) whenever you know the total amount (product) and the other number about equivalent parts (factor).\(^6\)

These understandings are very useful when there are equations or problems to solve. Students can reflect on many of these ideas even as they encounter the combinations of arithmetic. Without such “deep meanings” for the operations, students tend to merely react to the symbols they see and do not make the needed connections conceptually. For example, the student who sees \(56 + \square = 83\) may think “56 and 83; it is *plus* so you add the numbers.”

Error patterns are sometimes learned by students who lack adequate understanding of what the operations mean and how they are related. Consider the papers that follow. Can you decide how these students determined the unknown in each case?

These students would find it helpful to apply the understandings listed above.

---

**Darlene**

A. \(\square - 384 = 126\) \hspace{1cm} \(\square = 258\)
B. \(\square \times 13 = 260\) \hspace{1cm} \(\square = 20\)
C. \(\square \div 40 = 20\) \hspace{1cm} \(\square = 800\)

---

**Ronnie**

1. \(65 \div 13 = \square\) \hspace{1cm} \(\square = 5\)
2. \(\square \div 12 = 36\) \hspace{1cm} \(\square = 432\)
3. \(17 \times \square = 68\) \hspace{1cm} \(\square = 4\)
4. \(60 \div \square = 30\) \hspace{1cm} \(\square = 1800\)
5. \(24 \times 8 = \square\) \hspace{1cm} \(\square = 192\)
6. \(90 \div \square = 15\) \hspace{1cm} \(\square = 1350\)
ATTAINING COMPUTATIONAL FLUENCY

Admittedly, before we teach our students how to find sums, differences, products, or missing factors, we must teach when such numbers are needed. Students must understand the meanings of the operations in order to know which button to push on the calculator or which algorithm to use; that is, whether to add, subtract, multiply, or divide.

When a student does need to compute, there are actually four different ways to obtain the number: estimation, mental computation, paper-and-pencil algorithm, and calculator (or computer). As adults, we compute in all these ways; we use the method appropriate to the situation. Our students, if they are to be fluent in computation, also need to be able to compute in each of these ways; and they need practice in choosing the appropriate method for particular situations. They need to practice choosing the appropriate form of computation to use while solving varied problems in unfamiliar situations—including real-world contexts. Figure 3.1 illustrates how methods of computation fit within a problem-solving context.

When should each method of computation be used? This is a judgment that must be made in context, but the guidelines in Figure 3.2 may be helpful.

In Principles and Standards for School Mathematics, NCTM stresses the need for students at all levels of instruction to be able to use computational tools and strategies fluently and estimate appropriately. The term “computational fluency” reflects the ability to efficiently use different forms of computation as appropriate.17

When computation procedures are taught, a balance between conceptual and procedural learning is needed. If instruction focuses exclusively on following

FIGURE 3.1 Methods of computation chosen within a problem-solving context.

CONTEXT: Problem Solving
1. Understand the problem
2. Devise a plan
   This often involves understanding the meanings of the different operations on numbers.
3. Carry out the plan
   This often involves choosing the appropriate method of computation.

METHODS OF COMPUTATION

<table>
<thead>
<tr>
<th>Approximation</th>
<th>Exact Computation</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Estimation</td>
<td>• Mental computation</td>
</tr>
<tr>
<td></td>
<td>• Paper-and-pencil algorithm</td>
</tr>
<tr>
<td></td>
<td>• Calculator or computer</td>
</tr>
</tbody>
</table>

4. Look back

(Source: Based on George Polya, How to Solve It, 2d. ed. [Princeton, NJ: Princeton University Press, 1973], xvi–xvii.)
procedures and rote memorization, students’ habits of mind are likely to become less curious and creative in their approach to solving problems; if students memorize procedures by rote, they are less likely to remember them. When students “... have memorized procedures and practiced them a lot, it is difficult for them to go back and understand them later.”\(^\text{18}\) Students need instruction that is balanced—instruction that involves both conceptual and procedural learning.

That balance does include practice. Students need to practice reliable procedures and develop computational fluency if they are to become good problem solvers. As much as possible, our instruction in computation should be within real-world, problem-centered contexts, with isolated drills or games as supplemental instruction when required to enhance specific skills. Keep the focus on problem solving.

**TEACHING MENTAL COMPUTATION**

Students and adults use mental computation during daily living more often than they use written computation. Truly, mental computation is an important skill for students to learn. It is also a “natural stepping-stone to developing written computation and computational estimation.”\(^\text{19}\)

Mental computation is concerned with exact answers, but there is no set procedure for computing a particular operation such as addition. Instead, strategies involving known concepts and principles are applied thoughtfully, flexibly, and creatively within particular situations. For example, to mentally compute the sum $98 + 99$ an individual may choose to apply knowledge that $98$ is also $100 - 2$ and $99$ is $100 - 1$, and reason that their sum is $200 - 3$ or $197$. (This is actually easier than doing the paper-and-pencil procedure “in your head.”) As our students come to understand an operation like addition and how a number can be renamed as a difference, they should be given opportunities to apply this knowledge by computing mentally.

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**FIGURE 3.2** Guidelines for selecting the method of computation.

<table>
<thead>
<tr>
<th>Use</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimation</strong> when an approximate answer is sufficient, for example when the question is: About how many?</td>
</tr>
<tr>
<td><strong>Mental computation</strong> when an exact answer is needed, and it can be readily computed mentally by using known facts and principles: for example, basic facts, multiplying by powers of ten, and distributivity, as in computations like $7 \times 604$ or $6 \times 98$.</td>
</tr>
<tr>
<td><strong>Calculator or computer</strong> when an exact answer is needed, a calculator or computer is readily available, and computation would be quicker than using other methods, as in $508,032 \div 896$.</td>
</tr>
<tr>
<td><strong>Paper-and-pencil</strong> when an exact answer is needed and other methods are not appropriate or available.</td>
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Use

- **Estimation** when an approximate answer is sufficient, for example when the question is: About how many?
- **Mental computation** when an exact answer is needed, and it can be readily computed mentally by using known facts and principles: for example, basic facts, multiplying by powers of ten, and distributivity, as in computations like $7 \times 604$ or $6 \times 98$.
- **Calculator or computer** when an exact answer is needed, a calculator or computer is readily available, and computation would be quicker than using other methods, as in $508,032 \div 896$.
- **Paper-and-pencil** when an exact answer is needed and other methods are not appropriate or available.
Mental computation strategies can often be learned and practiced as warm-ups before math lessons. We should not wait until after we teach paper-and-pencil algorithms. Mochón and Román conclude from their research that it is “. . . wise to develop strategies of mental computation before or simultaneously with introduction of the formal algorithms.”

Instruction in mental computation often applies principles of numeration like place value and properties of operations like commutativity, associativity, and distributivity. Many specific strategies can be taught to facilitate mental computation. Examples include the following.

- Find pairs of numbers that add to one or to 10 or to 100. For example, think of 45 + 76 + 55 as 45 + 55 + 76, then 100 + 76 is 176.
- Find a more useful name for a number. For example, 28 + 56 is the same as \(30 - 2\) or 84.
- Do the operation “in parts.” Distributivity can often be used. For example, 4 \(\times\) 7 is the same as 4 \(\times\) (5 + 2), and 20 + 8 is 28 (a partitioned array will help).
- Use numbers that are multiples of 10 and 100 and 1000. For example, 4 \(\times\) 298 is the same as 4 \(\times\) 300 – 2, and 1200 – 8 is 1192.

Instruction in mental computation can help many of our students develop a flexible approach to computation. They will be less likely to limit a needed computation to a particular procedure.

**Teaching Students to Estimate**

Our students need to learn to estimate not only to solve problems which do not require an exact number, but also to make sure results are reasonable when performing exact computations. A proper emphasis on estimation will eliminate much of the need for future corrective instruction.

Instruction in estimation must begin early and occur often. Students who estimate well are thoroughly grounded conceptually. But any student who thinks that 27 is closer to 20 than 30 will have difficulty estimating, as will the student who does not understand that \(\frac{2}{5}\) is almost 1. Students need a robust number sense in which numeration concepts are understood and applied, and number combinations are easily used as are compensation principles and other relational understandings.

Attitudes toward estimation are also important. Typically, students believe “there is only one correct answer,” but when estimating, there are only reasonable answers—and some answers are more reasonable than others. Our students must learn to recognize when an estimate is all that is needed, and they must feel free to use terms like almost, a bit more than, about, a little less than, and in the ballpark.
The ability to estimate incorporates varied mental computation skills, any one of which may require instruction. Included among such skills for whole numbers are:

- Adding a little bit more than one number to a little bit more than another; adding a little bit less than one number to a little bit less than another; and, in general, adding, subtracting, and so on, with a little bit more than or a little bit less than.
- Rounding a whole number to the nearest ten, hundred, and so on.
- Multiplying by ten, and by powers of ten—in one step.
- Multiplying two numbers, each of which is a multiple of a power of ten (e.g., $20 \times 300$). This should be done as one step, without the use of a written algorithm.

With fractions and decimals less than one, estimation often involves using benchmarks; determining if a particular number is closer to zero, one-half, or one.

When possible, we should teach estimation informally in the context of problem solving. Here are a couple of examples:

- Problems involving the total cost of items purchased are reasonable to estimate because the buyer needs to know how much money to have at hand for the cashier.
- The purchase of a discounted item also requires an estimate of actual cost and the amount of money needed for the cashier.

Several estimation strategies can be taught with the hope that students will use them flexibly as appropriate. Strategies listed by Reys can be explained and illustrated as follows.21

- **Front-end strategy.** Numbers are rounded to greater place values and then the operation is performed. For example, for the sum $678 + 724$ think $700 + 700$, and the estimate is about 1400.
- **Clustering strategy.** The numbers are close in value and an “average” is obvious. For example, the average of 24,135 and 23,687 and 25,798 is about 24,000.
- **Rounding strategies.** One strategy is using upper and lower bounds for multiplication. For example, the product for $63 \times 89$ is between $60 \times 80$ and $70 \times 90$; it is between 4800 and 6300, possibly near 5500.
- **Compatible numbers strategy.** Alternative numbers are selected because they can be computed mentally. Consider the example $6128 \div 9$. Think $6300 \div 9$ and the estimate is almost 700.
- **Special numbers strategy.** It is noted that given numbers are about as much as particular benchmark numbers (0, 0.5, 1, 10, 100, and so on). Consider the example $\frac{7}{8} + \frac{8}{9}$. Think $1 + 1$ and the estimate is a little less than 2.
One way to provide practice with estimation is to present students with a problem and several possible answers. Students can then use estimation to choose the answer that is most reasonable. Here is a problem with possible answers:

A $1,495 large-screen television set has been discounted 20%.

Which cost is the most reasonable estimate?

$1,000 $1,100 $1,200 $1,300 $1,400

Also, if students are given numerical expressions such as the following, they can be asked to list the expressions in order from least to greatest. For the following expressions, students would list the letters in order from least product to greatest product.

a. 89 × 102   b. 75 × 98   c. 48 × 320   d. 8.9 × 10.2

Parents can encourage practice with estimation by having children estimate amounts while cooking and distances between places.

In general, our students will become more and more able to determine when an answer is reasonable as they gain the habit of asking if the answer makes sense, and as they develop a more robust number sense. Students who habitually consider the reasonableness of their answers are not likely to adopt incorrect computational procedures.

**TEACHING STUDENTS TO USE CALCULATORS**

When solving a problem, sometimes the sensible choice for computation is a calculator. Occasionally students think that using a calculator is cheating, so we need to make sure our students experience and think about a calculator as a viable choice for computation.

Of course there are times during instruction when calculators should be set aside—when mental computation or estimation is the focus, for example. Usually, when students are being encouraged to invent paper-and-pencil procedures or when they are studying particular conventional algorithms, calculators are not used.

Often, students who are free to use calculators can solve a more extensive range of problems, and they can approach these problems earlier. For example, they can use real-world problems based on situations reported in the local newspaper—even when they have not developed paper-and-pencil procedures for the required computations.

Though there may not yet be a consensus among mathematics educators about how calculators are to be used at every level of instruction, research does suggest that “calculators should be an integral part of mathematics instruction including the development of concepts and computational skills.” A meta-analysis of 54 research studies concluded that “calculator use did not hinder the development of mathematical skills” and that “students using calculators had better attitudes toward mathematics.”
As we teach our students how to use calculators, we must be careful not to focus exclusively on answers. Rather, we must focus on the thinking processes of students and their application of concepts and mathematical principles. Reasoning through a two-step problem, for example, requires much more than entering numbers in a calculator.\textsuperscript{24}

We must help our students understand what a calculator can do and how to use it, then give them opportunities to use calculators throughout the mathematics program. When used appropriately, calculators can even help students develop number sense and mental computation skills and help them understand numeration concepts and the meanings of operations. When our students use calculators, we should talk with them from time to time and have them explain what they are doing and why specific choices are made.

A calculator is not simply an alternative to paper-and-pencil procedures; it can help our students learn those procedures. For example, a calculator can be used to focus attention on one step within an algorithm. Calculators can also be used to practice estimating quotients. Consider the following game.

- Provide several examples similar to 83,562 \div 36 or 17,841 \div 892; then have students agree together on one example.
- Each person estimates the answer to that example and writes it.
- One student determines the exact answer with a calculator.
- Players score one point if they have the correct number of digits in their estimate; and they score two points if they have the correct number of digits and also the first digit is correct.\textsuperscript{25}

Of course a calculator can be used to reinforce underlying concepts and procedures—especially numeration concepts. For example, students can practice naming what some call the “product value of a digit” (face value \times place value). Consider these instructions for a game.

Everyone enter “1111” in your calculator.
I have a set of cards; each has one of the digits 2 through 9 on it.
When I draw a card, use addition or subtraction to change a one on your calculator to the number shown on the card. You decide which one to change.
Then I will draw another card and you can change another of your ones.
After four cards are drawn and you have changed all four ones, we will see who shows the greatest number on their calculator.

Basic multiplication products can be generated by using the repeat function of calculators.

\[6 \times 7 = ?\] Think of \(6 \times 7\) as 6 sevens.

Key \[
\begin{array}{cccccc}
7 & + & 7 & = & = & = \\
\text{counting} & 2, & 3, & 4, & 5, & 6,
\end{array}
\]
We can even use calculators to provide immediate feedback when students practice recalling number combinations.

As noted by Huinker, a calculator can even be used with kindergarten and first grade students to explore numerals, counting, number magnitude, and number relationships.

Indeed, a calculator has many uses—but its limitations must also be demonstrated. For example, it takes more time to multiply by a power of ten on a calculator than to perform the multiplication mentally.

**TEACHING PAPER-AND-PENCIL PROCEDURES**

Conventional paper-and-pencil algorithms involve more than procedural knowledge; they entail conceptual knowledge as well. Many of the instructional activities described in this book are included because of the need for conceptual understanding. Students are not merely mechanical processors, they must be involved conceptually when learning and using paper-and-pencil procedures.

Even so, it must be recognized that as a student uses a specific paper-and-pencil algorithm over time, the procedure becomes more automatic. Students gradually use less conceptual knowledge and more procedural knowledge while doing the procedure, a process researchers sometimes call “proceduralization.”

We must be careful not to introduce a paper-and-pencil procedure too early; we need to be especially careful not to use direct instruction about steps in a procedure too soon. Frequently, we can introduce an algorithm with a verbal problem and challenge our students to use what they already know to work out a solution—even if their prior knowledge is quite informal. When we let students use their own informal techniques initially, we will find that some students know more than we thought! Others will creatively use what they already know and the manipulatives we make available. By beginning this way, we will help students relate the algorithm we are teaching to their prior knowledge. We may even want to have a group of students investigate different ways of finding a sum, a product, and so on. Students should stay with the investigation long enough for several alternatives to be developed and shared; they could even write about their experiences.

Students who are permitted to work out solutions using informal knowledge before they are taught a specific computational procedure sometimes develop “invented” paper-and-pencil procedures. Place a high value on all invented procedures and the creativity involved; say something like, “That’s great! Why does it work? Will your procedure give you the correct number every time? How can you find out?” Invented algorithms are often evidence of conceptual understanding. Invented computational procedures are not always efficient, but they are correct procedures if they always produce the number needed.

**Instruction in Grades 1–2**

Mathematics educators agree that in Grades 1–2 they want students to understand numbers and how numbers are related to one another. They want students to be
able to represent quantities and to understand addition and subtraction and how those operations are related to one another. Fluency with addition and subtraction number combinations is also a goal. There is less agreement on the place of computational procedures in the mathematics curriculum for Grades 1–2.

Should the emphasis be on students inventing procedures or on students learning conventional algorithms? In Principles and Standards for School Mathematics, NCTM includes the following statements in its discussion of standards for number and operations for Pre-K-2 (emphases added):

- Students learn basic number combinations and develop strategies for computing that make sense to them when they solve problems with interesting and challenging contexts.
- Through class discussions, they can compare the ease of use and ease of explanation of various strategies. In some cases, their strategies for computing will be close to conventional algorithms; in other cases, they will be quite different.
- When students compute with strategies they invent or choose because they are meaningful, their learning tends to be robust—they are able to remember and apply their knowledge.
- Students can learn to compute accurately and efficiently through regular experience with meaningful procedures. They benefit from instruction that blends procedural fluency and conceptual understanding. . . . This is true for all students, including those with special educational needs.

Clearly, NCTM recommends that we focus on meaningful learning. In the earliest grades this can involve both invented procedures and conventional algorithms, but the stress should be on thinking and on procedures that make sense to students.

Trafton and Thiessen argue that children need to learn a wide variety of strategies for problem solving and computation; they emphasize that such strategies are not learned at one specific time or in a single lesson. Children learn strategies for solving problems and computing on their own timetable and not ours.

When our students invent procedures and record them, we can ask questions that point toward more efficient refinements in the procedures students are developing. Invented algorithms can often be further developed into conventional algorithms, but some question the need to do this if the invented algorithm is correct and a reasonably efficient procedure. It is probably true that in Grades 1–2 we should encourage students to invent procedures but also be open to helping them learn about conventional algorithms as warranted. Curcio and Schwartz argue for a balance.

Some would delay teaching conventional procedures. In their proposed sequence for teaching computation, Reys and Reys suggest that conventional algorithms for addition and subtraction of whole numbers be delayed until Grade 3, and taught then only if students have not already developed ways of computing that are efficient.
Developing number sense should be the primary focus of instruction in Grades 1–2, for well-developed number sense provides a foundation for computational fluency. Conventional algorithms, whenever they are taught, should be yet another context that makes sense for investigating mathematics.

Conventional procedures can be taught so students understand the algorithms. When this is done, instruction typically involves the use of manipulatives.

**The Role of Manipulatives**

Teaching paper-and-pencil procedures should not be merely a demonstration of “how to do it” accompanied by an explanation. Such attempts are inadequate for most students, especially young children. They do not result in conceptual learning, and very often they do not even result in procedural learning.

Visual, tactile, and kinesthetic experiences provided through manipulatives can help our students better understand the numbers involved, their numerals, and the operation itself. When manipulatives are used—whether with an invented procedure or a more conventional algorithm—then the steps in the procedure are apt to make sense. Our students are more likely to gain confidence they can learn and do mathematics.

As our students use manipulatives to find needed sums, differences, products, and quotients, we must make it clear that we value their solution attempts. We can challenge students to provide evidence that what they are doing always works. Then, if we choose to guide them toward a paper-and-pencil procedure, we simply encourage them to record what they are doing. Ideally, their manipulations can be recorded on paper “step-by-step.”

Game-like activities using a pattern board are sometimes used as a bridge between students informally working out solutions with manipulatives, and more direct instruction in conventional algorithms. Such activities for addition, subtraction, and division of whole numbers are described in Appendix C. The pattern board serves as an organizing center; and a step-by-step record of what is done on the board turns out to be a conventional algorithm that can be seen as a mathematical representation of what was observed. The paper-and-pencil computation procedure makes sense to students because they have observed relationships and patterns; they have a visual referent for the algorithm itself.

**Developmental Instruction**

Developmental instruction in conventional computational procedures must be distinguished from corrective instruction, which is discussed briefly in the section that follows. The term *developmental instruction*, as used here, is the initial sequence of instructional activities over time that enables students to understand, execute, and gain skill in using particular algorithms. *Corrective instruction* follows developmental instruction whenever a student has not learned a correct procedure; for example, a student may have learned an error pattern during initial instruction. Careful developmental instruction seeks to help students learn algorithms *without* learning error patterns.
Before we teach our students to compute on paper, we must make sure they are able to represent quantities with appropriate notation. Also, they need to be able to make suitable exchanges with manipulatives; when working with whole numbers, chip-trading activities can help our students develop these abilities.

As we introduce a particular algorithm and continue to provide instruction, we must engage our students in thinking—not in mindless copying and repetition. While our students are first learning a computational procedure, they need to make mental connections and build the procedure in their own heads.

We can begin by having our students use manipulatives to find solutions to problems; base-ten blocks are frequently used for addition and subtraction of whole numbers, and fraction parts are often used for addition and subtraction with fractions. Our students need to use manipulatives initially to solve problems—whether invented procedures are stressed, game-like activities are incorporated, or the instruction has a conventional algorithm as its goal. It is important that students begin with manipulatives, then reflect on what they have done with the materials.

Eventually, a step-by-step record of manipulations and thinking is written with numerals. If our goal is to teach a conventional algorithm, we must keep that procedure in mind as we guide the recording; the written record of manipulations can become the algorithm itself. When our students are comfortable with this process, they will be able to visualize the manipulatives (but not actually handle them) as they write. In some cases, if students are to develop a more efficient algorithm, possibly a conventional algorithm, they will need to shorten the written record. We may want to say, “Mathematicians like to write fewer symbols whenever they can.”

Instruction will be meaningful if it is done within a problem-solving context and the algorithm is developed as a step-by-step record of observations. The computational procedure will make sense to our students because it is a record of what they have actually seen. Typical elementary school students move very gradually from making sense through manipulatives to making sense through mathematical reasoning. Any student experiencing difficulty while attempting to learn a computational procedure may need to work more directly with manipulatives for a while.

Admittedly, there are algorithms that cannot be developed as a record of observations—especially in the middle grades. Sometimes these procedures can be introduced as a short cut. For example, the conventional algorithm for dividing fractions can be developed by reasoning through a rather elaborate but meaningful procedure involving complex fractions, applying the multiplicative identity and the like, then observing a pattern. The obvious implication of the observed pattern is that most of the steps can be eliminated; merely invert the divisor and multiply.

Teachers and curriculum designers are faced with the question, “When should different stages for an algorithm be introduced?” Traditionally, a rather rigid logical sequence was followed in textbooks; for example, addition of whole numbers with no regrouping was taught well before addition with regrouping. But when we teach computation in the context of solving problems, the problems of interest do not always occur within that traditional sequence. This should not deter us and our students from exploring solutions for interesting problems that will lead to more generalized written procedures. Usnick found that initial teaching of
the generalized procedure for adding whole numbers (regrouping included) led to comparable achievement and effective retention.\textsuperscript{33}

### Carry? Borrow? Regroup? Rename?

Which term is appropriate when adding or subtracting whole numbers? Obviously, “carry” and “borrow” are misleading mathematically, though the terms are often used. They may promote mechanical manipulation of symbols instead of a procedure that makes sense to students.

The term “regroup” is appropriate when manipulatives for a quantity are grouped differently. The term “rename” is mathematically correct; the quantity is actually given a different name. For example, when computing $273 - 186 = 2 \text{ hundreds} + 7 \text{ tens} + 3 \text{ ones}$ is renamed as $2 \text{ hundreds} + 6 \text{ tens} + 13 \text{ ones}$.

Other terms that may cause students to focus only on a procedure are “reduce,” “cancel,” and “invert.” Make sure students understand the concepts involved.

When using manipulatives to teach a conventional algorithm, the critical step is progressing from manipulatives to written symbols. This is why a step-by-step record is helpful. The resulting record or algorithm must make sense to our students if they are going to do more than push symbols around on paper.

We must be sure our students’ paper-and-pencil procedures are correct before we encourage them to make the procedures automatic. When algorithms are correct, a certain amount of practice is required for the procedures to be remembered and used effectively; but practice with paper-and-pencil procedures needs to be planned carefully—as does practice for all methods of computation. NCTM notes in *Principles and Standards for School Mathematics*:

Practice needs to be motivating and systematic if students are to develop computational fluency, whether mentally, with manipulatives materials, or with paper and pencil. Practice can be conducted in the context of other activities, including games that require computation as part of score keeping, questions that emerge from children’s literature, situations in the classroom, or focused activities that are part of another mathematical investigation. Practice should be purposeful and should focus on developing thinking strategies and a knowledge of number relationships . . .\textsuperscript{34}

Continuing diagnosis is very important. When we say, “Tell me something about this,” we help our students develop the ability to communicate mathematical ideas—even as they give us diagnostic information. We must keep our “diagnostic eyes” open throughout instruction. However, it is also important to not over-test students—especially at-risk students. Although diagnosis should continue throughout instruction, we should never limit instruction to assessment activities. Sometimes teachers fall into that trap.

Our written and oral responses to students’ written work in mathematics affect students—either positively or negatively. It is best to give an immediate
personal response to what the student is doing rather than a list of things to be done next time.

“I have no trouble reading your numerals.”
“Very interesting! How did you get your answer?”
“Did you think about your answer? Does it make sense?”

Sometimes, when teaching a specific algorithm, it is helpful to have a group of students analyze a completed example. It is important to emphasize thinking as students observe, describe, and hypothesize what was done. Students should discuss why it resulted in the correct number and try the procedure with different numbers. We may also want to ask some students to analyze incorrect computations, suggesting that they find and explain the errors.

Many of our students will make mistakes while learning to compute. Even so, mistakes can be an important, positive part of the initial learning process. Interestingly, teachers respond differently to errors in different cultures.

We have been struck by the different reactions of Asian and American teachers to children’s errors. For Americans, errors tend to be interpreted as an indication of failure in learning the lesson. For Chinese and Japanese, they are an index of what still needs to be learned. These divergent interpretations result in very different reactions to the display of errors—embarrassment on the part of American children, calm acceptance by Asian children. They also result in differences in the manner in which teachers utilize errors as effective means of instruction.35

Our students who make mistakes and experience failures need to feel that we accept their failures along the way as an expected part of learning.

Because the learning tasks are sizable, important, and difficult, students cannot expect to “get it right” at every point along the way. Failure is used in these instances as a step toward ultimate mastery and understanding. Thus, failure is a temporary, and admittedly often painful, part of the journey to success.36

Our attitudes toward errors are important. We should view them as opportunities for learning!

We must monitor our own expectations of students, making sure we do not assume particular individuals cannot learn. Even so, we do need to be alert to any perceptual difficulties a student may have. In order to respond to instruction, students must be able to observe and also envision the physical properties of digits: vertical versus horizontal elongation, straightness versus curvature, and degree of closure. And our students must be able to perceive attributes of multi-digit numerals—properties such as position of a digit to the left or right of another digit. Poor spatial ability may affect an individual’s capacity to respond to instruction emphasizing place value concepts.
Other students may find it difficult to respond to instruction because of language patterns. The syntax of English language expressions is often different from the structure of mathematical statements, and we complicate the situation by using different but equivalent language expressions for the same concept. For example, \textit{twelve minus four} and \textit{four from twelve} express the same mathematical concept.

Teaching conventional computational procedures requires thorough developmental instruction; each student moves through a carefully planned sequence of learning activities. The amount of time needed for each type of activity will vary from student to student; and for any individual, the pace will likely vary from day to day. If we are to lessen the likelihood that students learn patterns of error, we will have to resist the temptation to cover the text or the curriculum guide by completing two pages a day or a similar plan. Careful attention will have to be given to ideas and skills needed by each student in order to learn the concept or algorithm under study.

We can teach in a way that makes the adoption of erroneous procedures unlikely!

\textbf{Corrective Instruction}

Corrective instruction may be necessary whenever one of our students has not been able to learn a computational procedure that produces correct answers in an efficient way. Corrective instruction must be built on a careful diagnosis of what the student has and has not learned and on what the student can and cannot do.

The student may have acquired a simple misconception that can be corrected with focused instruction. Sometimes the student is not adequately grounded in the concepts and principles needed to understand the algorithm, then corrective instruction must begin by teaching foundational concepts and principles rather than the computational procedure itself.

For students who adopt error patterns, it is often wise to redevelop computational procedures as careful step-by-step records of observations while using manipulatives. Hopefully, this will help the student who has been pushing symbols around in a rote manner to make sense of his record.

Students need feedback that not only tells them which examples are correct but which also assists in obtaining correct answers. Corrective feedback can take many forms. It can be presented orally along with personal comments that express confidence in his ability to learn, or it can be written on the student's paper. All too frequently, teacher feedback does not include corrective feedback; papers are merely scored and students are asked to rework the examples. When appropriate, we need to write on papers, notes providing personal, corrective assistance.

\textbf{Students with Learning Disabilities}

As we reflect on ways we can adapt instruction to the individual needs of students in our classrooms, we need to consider how we will differentiate instruction for students with learning disabilities. They have diverse specific needs, and if we are to help them, we must be ready to intervene with a variety of strategies.
What difficulties do students with learning disabilities have in regard to learning mathematics? And specifically, how do those disabilities affect learning paper-and-pencil computation procedures?

Students with learning disabilities, as well as other students who experience limited success, often have attention problems: they have difficulty listening to directions, attending to all the steps, and completing their work. How can we encourage attending behavior? Bender's suggestions for fostering attention skills include the following:37

• Display three or four positively stated classroom rules.
• Post a daily class schedule, even for short departmentalized periods.
• Train on class cues. A bell can be rung for attention. Cue cards or charts can remind students how to begin a lesson (get out your book, etc.)
• Keep desks clear. Students should only have on their desks texts and materials needed for the lesson.
• Use peer buddies to make sure both are ready to begin the next activity or lesson.

There are other characteristics common among students with learning disabilities that also make it difficult for these students to learn mathematics. Steele describes the following:38

• Memory deficiencies—difficulty learning facts or remembering the correct sequence of steps for a particular skill.
• Auditory processing problems—difficulty understanding oral explanations of content and vocabulary.
• Visual processing problems—losing his or her place, confusing numbers such as 17 and 71, copying inaccurately, working problems in the wrong direction, and lining up work incorrectly with respect to place values.
• Abstract reasoning deficits—struggling more than usual with word problems and new concepts, and “shutting down” mentally when they see tasks they associate with failure.
• Organizational deficiencies—difficulty in selecting and using appropriate strategies. Sometimes these students master one strategy, then apply it regularly, even when inappropriate.

How should we structure our lessons when teaching students with learning disabilities? Bender lists these suggestions:39

• Provide clear directions, particularly during transitions.
• Develop alternative instructional activities—e.g., two worksheets that present the same material at different levels, and alternative assignments that cover the same material.
• Plan frequent breaks. Take “stretch breaks” (about 30 seconds) every 15 minutes or so.
• Use physical activities. For example, movement can be involved when students learn basic combinations.
• Use clear worksheets. Make sure they are not cluttered and do not contain distractions.
• Decrease task length. A longer task can be presented as a set of shorter, doable tasks.
• Develop alternative assessment tasks. As an alternative to paper-and-pencil assessment tasks, for example, have the student orally tell you how to do the computation, or show you the procedure with base ten blocks.

If we are to provide effective instruction for those of our students who have learning disabilities, our instruction will need to be modified in many ways. Other suggestions are noted by Steele:

• Present advance organizers. (The game board activities described in Appendix C are examples for paper-and-pencil computation procedures.)
• Review prerequisite skills or concepts. This should be done no matter how long ago they were taught.
• Model procedures enough times. Demonstrate a procedure slowly and repeatedly until it is clear.
• Use mnemonic strategies. These may be acronyms.
• Use peer tutoring. This may be especially effective for improving calculation skills.

Those who will teach mathematics to special needs students will want to read the October 2004 issue of Teaching Children Mathematics 11(3), a focus issue on teaching mathematics to special needs students.

**Developing Mathematical Vocabulary**

Concepts and principles are named with words and expressed with words and symbols. And it is possible that many of our students will stumble over the specialized vocabulary associated with mathematics. A term may not be associated with the appropriate concept, or a concept understood may not be given the appropriate name. From time to time, something akin to the overgeneralizing and overspecializing illustrated in the previous chapter is likely to happen as our students learn concepts and procedures. The result is muddled communication and confused thinking.

For younger students, a general teaching strategy is to introduce words and symbols as a way of describing and recording what students have already observed and know informally. They use words informally; their math ideas are not expressed in final form initially—but that is also true of professional mathematicians. If we begin with students’ very informal language—often the way students
describe situations—we may be able to introduce more precise terminology in ap-
position to the informal language, gradually dropping the informal language.

For example, the term “addend” can be developed while students are exam-
ining sets separated into two subsets. The number of items in each subset and the
total number of items are recorded. At first, the number of items in each subset is
informally called the “number for the part.” Later, something like “the number for
this part, or addend, is five,” is appropriate. Eventually this leads to expressions
like. “What is the other addend?” Often older students can simply be told, “We
call the number for a part (or subset) an addend.”

Some people seem to think that mathematics is culture free because it in-
volves numbers. But learning mathematics involves language, and, obviously, that
is not culture free. For example, in Japanese, the fraction called “one fourth” in
English is yon bun no ichi; the denominator is read first, and the expression means
“one of the four partitions.” The Japanese more clearly points to a part-whole in-
terpretation of a fraction.43

We should not be in a hurry for our students to use precise mathematical
terms. Steele cautions that “we should not move students too quickly toward new
mathematical language without giving them the opportunity to explore, investi-
gate, describe, and explain ideas.”44

When our students are learning mathematical vocabulary, they need to make
connections with concepts and terms they already know. It is often helpful for
them to connect with root meanings and related words. For example, the “nom”
in denominator means “to name.” A denominator names a fraction; it indicates the
kind of fraction, the size of the parts. And “num” in numerator suggests “num-
ber.” A numerator tells the number of parts.

A concept web or a simple diagram may help those who are visually ori-
ented. Figure 3.3 shows that the relationship between factors and multiples can be
illustrated.

We may want to have students focus on mathematical vocabulary by writing
on cards the terms they encounter. We can have students place the cards in cate-
gories such as the following—a few of the categories of difficulty described and il-
lustrated by Rubenstein and Thompson.45

- Words that are also used in everyday English, but with different meanings
  (e.g.: foot, reflection)
- Words with more than one mathematical meaning
  (e.g.: square, second)
- Words which are homonyms with everyday English words
  (e.g.: sum, arc, pi)
- Words or patterns with irregularities
  (e.g.: four but forty; one fourth, one-third then one-half)

Sometimes we can make mathematical vocabulary clearer if we employ one of
the key concepts of school mathematics—the idea that a number has many names.
For instance, the term “factorization” is sometimes confusing. Factorizations are
particularly useful names for numbers, especially factorizations that consist solely of prime numbers; they are called “prime factorizations.” Changing a fraction to an equivalent fraction involves another use of the idea that a number has many names. This process involves finding a different name for the same number, a name that is more useful for the purposes at hand.

As our students acquire and use mathematical vocabulary, we need to support classroom talk about mathematics that moves away from very specific contexts and moves toward applications of mathematical ideas in varied contexts.


**Talking and Writing Mathematics**

Both talking mathematics and writing mathematics are teaching strategies that can be used while teaching computation. They enhance learning by involving our students in expressions of meaning and by giving them opportunities to relate the everyday language of their world to math language and to math symbols. They also provide opportunities for integrating mathematics with other subject areas.
Of the two strategies, the writings of students are typically more reflective. When students write prose as a part of instruction in mathematics, they sometimes demonstrate understandings that are not adequate, or they use terms and symbols incorrectly. Be sure to read their writings diagnostically.46

Talking mathematics involves more risk taking on the part of students than does writing mathematics, at least at first. One activity that involves talking mathematics is for students to conduct an interview with an older adult to learn about specific paper-and-pencil algorithms they were taught and use now. Part of such an interview follows.

STUDENT: Grandpa, please subtract this for me. And think out loud so I'll know how you are doing it.

\[
\begin{array}{c}
253 \\
-179 \\
\hline
174
\end{array}
\]

GRANDPA: OK. Nine and four is 13. Write four here and put the one under the seven.

\[
\begin{array}{c}
253 \\
-179 \\
\hline
174
\end{array}
\]

STUDENT: (later) That's different! Does it always work? . . . Why does it work? . . .

When students explain mathematical topics in a journal, it can be considered an “academic learning log.” Learning logs serve not only as records of student learning, but they also help students clarify their thoughts.47

Journal writing can be done for different audiences. It should be first-person writing that focuses on something related to mathematics or learning mathematics. Often, journals are part of a written dialogue with the teacher in which students express how they feel about mathematics, what they know with confidence, when they believe that knowledge can be used, and questions they may have.

We may want our students to keep two journals: one to write what they know or have learned, and another to tell how they feel about specific experiences in mathematics. Journal writing can help students focus on the topic at hand so they can ask appropriate questions; students who do not like to ask questions in class may be more likely to write their questions.

Journal writing is sometimes stimulated by prompts that we provide. Examples of appropriate prompts include: “I learned ______.” “I was pleased that I ______.” “I am most proud of ______.” And “I wish ______.”48
Journals are not the only way to write mathematics. These activities help students focus on what they are learning in mathematics.

- Write out definitions in your own words.
- In your own words, write out the most important information about finding equivalent fractions.
- Write test questions about what you studied today.
- Describe any places you had difficulty and how you worked it out.
- Write questions about what you do not understand. (Merely writing them may help reduce anxiety.)
- Respond to: “What I learned from my mistake.”
- Use a Solve and Comment Page for practice. (See Figure 3.4.)

Some of the following activities require students to produce written (or oral) explanations related to computation.

- Write how you would teach your cousin to add decimals.
- Write a letter to a student who was absent, and explain what is most important to understand about multiplying decimals.
- Use a hand puppet to explain to your younger sister how to subtract two-digit numbers.
- Make a poster or a bulletin board that explains how to divide whole numbers.

Students, individually or in groups, can respond to computations others have completed by talking and writing mathematics; both correctly and incorrectly.
worked-out examples can be included. Following is an example of a task that ac-
companies a paper with an error pattern similar to those illustrated in this book.

Score, analyze, and discuss each example.

For incorrect examples, describe in writing:
• what was done incorrectly,
• what should be done instead and why, and
• illustrate a correct procedure.

There are many occasions for talking and writing mathematics. Here is a list of additional things we can do to have our students write mathematics.

• Before a lesson, have students write in their journals to describe what they expect to happen. Later, students can write how they felt during the lesson.
• Have students do reflective writing to prepare for a discussion.
• Encourage students to write a math autobiography. You can stimulate similar writings with sentence starters like these:
  My best experience with math was _______.
  I liked math until _______.
  Math makes me feel _______.
  If I were a math teacher, I’d _______.

• Have students write word problems of their own. You can supply the data, or they can supply their own.
• Have students write and solve word problems for which a particular computational procedure would be appropriate.
• Let students write a story that teaches a math concept. For example, a young child could write a story to teach “the number 7” or “place value.”
• Encourage students to write a report or a book on a mathematical topic. Examples include numeration systems and algorithms used by different peoples.

**Using Graphic Organizers for Instruction**

Graphic organizers can enhance much of our instruction in mathematics, whether the organizers are used for regular developmental instruction or intervention with students experiencing difficulty. When our students learn new ideas, graphic organizers can serve as patterns for organizing and representing relationships: they can also serve as a means of assessing student learning. Whenever we use graphic organizers to provide the support needed for a student to succeed with challenging work, we are providing scaffolding for further learning.

Many types of graphic organizers can be incorporated into the teaching of mathematics. Here are some examples.
Diagrams. Many of the ways students represent and explain relationships are called “diagrams;” these are structural representations. A number line is one example.50

Drawings. Children can draw pictures of things their family buys at the grocery store, complete with price tags, and use them in constructing problems.

Charts. Fraction cards like \( \frac{2}{6} \) and \( \frac{3}{6} \) can be sorted in columns headed Less Than a Half, Names for One Half, and Greater Than a Half.

Overlapping circles. Phrase cards like \( 00 \).99 and \( 8 \cdot 8 \) and fraction cards can be placed in two overlapping circles, one labeled Fractions and the other labeled Names for One.

Webs and concept maps. Relationships among operations and among numbers are often shown with concept maps.51 An example of a web to be completed with cards by young children appears in Figure 3.5.

Graphs. Figure 3.6 illustrates a graph that students can make to record the results of their investigation regarding the number of basic addition combinations for each sum.

Grids. Grids typically display categories determined by two characteristics. Figure 3.7 illustrates a grid in which word problems on cards can be sorted by the operation required to determine the answer and by the most appropriate method for the computation.

Procedural maps. Figure 3.8 illustrates a procedural map for estimating the sum of two-digit numbers.

Flowcharts. Illustrations appear in figures 3.9, 3.10, and 3.11.

Number lines have uses that go beyond basic number and numeration work. For example, when teaching our students how to use basic number combinations

**FIGURE 3.5** A web for students to complete by sorting cards.
for adding or subtracting with higher decade numbers, a pattern can be observed with number lines.

This sequence illustrates a repeated number line shift.

\[ 5 + 8 = 13 \quad 15 + 8 = \boxed{} \quad 25 + 8 = \boxed{} \]

FIGURE 3.6  Graph for number of basic addition combinations for each sum.

![Graph for number of basic addition combinations for each sum.](image)

FIGURE 3.7  Grid for sorting word problem cards by most appropriate method of computation and by operation needed to find the answer.

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Paper</th>
<th>Mental</th>
<th>Calculator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subtract</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multiply</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Divide</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As you sort word problems, assume you have paper and pencil at hand, and it would take you about two minutes to locate a calculator.
Flowcharts are of particular interest for teaching computational procedures. We can use concept maps and flowcharts for an overview of what will be studied, or as a way of summarizing what has already been taught. To use them to assess student understanding, have students write in empty blanks or boxes within the maps and charts.

To teach simple flowchart procedures, make a chart available for reference that shows the basic shapes and how they are used. Figure 3.9 is an example of such a chart. When teaching simple flowcharting, focus on procedures for simple

**FIGURE 3.8** Procedural map for estimating the sum of two-digit numbers.

Estimate the Sum of Two-Digit Numbers

Addition example → Round each number to nearest ten → Add

Round to nearest ten

Estimate the sum

\[
\begin{align*}
29 & \quad + \quad 32 & \quad = \quad 80 \\
18 & \quad + \quad 20 & \quad = \quad 30 \\
30 & \quad + \quad 30 & \quad = \quad 30 \\
\end{align*}
\]

**FIGURE 3.9** Basic shapes for flowcharts.

FLOWCHARTS

- **start, or stop**
- **do**
- **decide (yes, or no)**
FIGURE 3.10 An example of a flowchart.

Choose the Greatest Number

\[\begin{align*}
68 & \\
29 & \\
45 & \\
63 & 
\end{align*}\]

Start

Find the greatest place value

Find the greatest number in that place value

Compare the next smaller place value

Yes

Is there more than one number?

No

Circle that number

Stop

mathematical tasks. Figure 3.10 is an example of a flowchart developed to show how to choose the greatest of several whole numbers.

Sometimes teachers provide a flowchart in which all of the shapes are blank. They also give students each of the individual shapes which have instructions written in them. Students cut out the shapes and decide where to paste them on the blank flowchart.

After our students have learned how to make a flowchart, we can have pairs of students, and eventually individuals, create flowcharts for paper-and-pencil computation procedures. Figure 3.11 is a flowchart for adding fractions prepared by a fourth-grade boy. Make sure that student-generated flowcharts are tested; other students can follow the chart step by step to see if the chart is complete and accurate.

Near the end of the school year, some teachers have their students record computational procedures they have learned during the year by making flowcharts they can take with them to their new classrooms.
Ultimately our students need to understand the basic number combinations of arithmetic and be able to recall them. Initially they need to understand the operations, but they eventually need to be able to recall number combinations without resorting to inefficient procedures.

... students should know the “basic number facts” because such knowledge is essential for mental computation, estimation, performance of computational procedures, and problem solving. \(^{32}\)
The basic number combinations, or basic facts of arithmetic, are the simple equations we use when we compute. They involve two one-digit addends if they are addition or subtraction number combinations, or two one-digit factors if they are multiplication or division number combinations. Examples include the following:

\[
\begin{align*}
6 + 7 &= 13 \\
12 - 8 &= 4 \\
3 \times 5 &= 15 \\
27 ÷ 9 &= 3
\end{align*}
\]

Students study number combinations in the context of learning what the operations of arithmetic mean. For instance, addition can be thought of as an operation that tells us the total number in a set if we know the amount in each of two disjoint subsets. Multiplication can also be conceived as an operation that tells us the total amount whenever we know two numbers: the number of equivalent disjoint subsets and the number in each subset.

Initially we should let younger students approach individual number combinations as problems to solve, often presented as open number sentences like \(5 \times 4 = \square\). When our students are permitted to investigate these problems in cooperative groups, they build on each other’s informal knowledge.

We must emphasize thinking during the study of number combinations and help our students make connections between them. Combinations for different operations are often related (some would say they are “close kin”); for example, \(5 \times 7 = 35\) and \(35 ÷ 7 = 5\) both have 5 and 7 as factors, and they have 35 as the total amount or product. Instead of always asking students to find a single number as in \(6 + \square = 13\), we should frequently pose more open-ended questions.

When two numbers are added, the total amount (their sum) is 13. What might the two numbers be? How many of the pairs of numbers include a six? What are the solutions to \(\square + \square = 13\)?

For addition combinations, Threlfall and Frobisher recommend that we stress patterns that students can use to generate new information. Visual patterns, for example, can be constructed with counting cubes or Cuisenaire rods and related to sequences of number combinations. Also, basic addition and subtraction combinations that have 9 as the sum or total amount can be generated as follows.

\[
\begin{array}{ll}
9 + 0 &= 9 \\
8 + 1 &= 9 \\
7 + 2 &= 9 \\
6 + 3 &= 9 \\
5 + 4 &= 9 \\
4 + 5 &= 9 \\
3 + 6 &= 9 \\
2 + 7 &= 9 \\
1 + 8 &= 9 \\
0 + 9 &= 9
\end{array}
\]
Other patterns can be related to compensation principles. For example, when one addend is increased by a particular number and the other addend is decreased by the same number, the sum remains the same. As our students consider this principle—possibly in a learning center—they can select number combination cards to place above and below relationship indicators.

Similar activities can be planned for other principles, such as the fact that if, while one addend remains the same, the other addend is greater by a particular number, the sum will be greater by that number.

Before mastery activities are introduced, our students need to be taught thinking strategies and more mature ways to determine a missing number. Isaacs and Carroll describe strategies which can help students understand and recall basic number combinations; they classify strategies in terms of focus: on counting, on parts and wholes, and on derived facts. The following list is an adaptation of categories suggested by Isaacs and Carroll.

**Counting**
- Counting on
- Missing addend
- Skip counting
- Repeated addition

**Observing Parts and Wholes**
- Make ten, and so many more
- 6 + 7 = (6 + 4) + 3 = 10 + 3 = 13

**Deriving Combinations from Other Combinations**
- One/two more/less than a known combination
- Use a double
- Compensation
- Use the related addition combination
- Use the related multiplication combination

For 13 – 5 think, 5 + ? = 13
For 42 ÷ 6 think, 6 × ? = 42
Same addends or factors (commutative) addition, multiplication
Multiply in parts: rename, multiply, and then add or subtract. 56
Nines pattern: next multiplication

Kamii and Lewis state that a child who knows the sum of two single-digit numbers solidly can deduce quickly the part that is unknown when presented with the related subtraction combination, and they argue that “... we must de-emphasize fluency in subtraction in the first two grades and heavily emphasize addition.” 57

Sometimes when we think a student was merely careless while attempting to recall specific number combinations, the student actually attempted one of the strategies listed above; however, the strategy was incorrectly applied. Perhaps the student attempted to count on, but miscounted. We need to assume that incorrect recall is rarely due to carelessness and attempt to find out what is really going on. Remember, diagnosis should be continuous.

Mastery of the basic number combinations of arithmetic is the ability to recall missing sums, addends, products, and factors promptly and without hesitation. A student who has mastered 6 + 8 = 14, when presented with “6 + 8 = ?” either orally or in writing, will recall 14 without counting or figuring it out. When a student attempts to find the product of two whole numbers (such as 36 and 457) by using a paper-and-pencil procedure, lack of mastery of the basic multiplication combinations requires time-consuming and distracting ways of finding the product. Lack of mastery of number combinations also greatly hamper mental computation and the ability to estimate.

If an older student has not mastered the basic number combinations, she probably persists in using counting or elaborate procedures to find needed numbers. She may understand the operations, yet she continues to require the security of counting or other time-consuming procedures when computing. She probably does not feel confident to simply recall the number. If such a student is involved in extensive practice activity, she reinforces use of less-than-adequate procedures. What she needs is practice recalling the missing number. How can we provide an instructional environment in which students like this one feel secure enough to try simply recalling missing numbers?

Games provide the safest environment for simple recall; when playing games, someone has to lose. The teacher always seems to want “the correct answer” but in a game it is acceptable to lose at least part of the time. The competition in a game encourages a student to try simply recalling the number combination.
Further, games often make possible greater attending behavior because of the materials involved. For instance, a student who rejects a paper-and-pencil problem such as “6 + 5 = ?” because it is a reminder of failure may attend with interest when the same question is presented with numerals painted on brightly colored cubes which can be moved about.

Obviously, what is intended is not an arithmetic game modeled after an old-fashioned spelling bee designed to eliminate less able students; nor is it a game designed to put a student under pressure in front of a large group of peers. The best games will be games involving only a few students, preferably students with rather comparable abilities. In such games, a student can feel secure enough to try simple recall. We should choose games that provide immediate or early verification; students should learn promptly if they recalled correctly. Commercial games are available, but games can be made using simple materials—many of which are already in the classroom. Our students are quite capable of making up games and altering rules to suit their fancy when they are encouraged to do so. A homemade game using a mathematical balance would provide immediate verification for each student's response (Figure 3.12). Kamii and Anderson describe how they made and used games involving basic multiplication combinations in a third-grade classroom, and they note that some of their games are adaptations of other games, even commercial games.58

Games are also useful for retention once the number combinations are mastered, and research suggests that relatively infrequent use of games can maintain skill with number combinations.59

When a particular basic fact is difficult for a student to recall, teachers sometimes use a combination of large visual presentation and muscle movement to help the student. For example, if the basic fact $8 \times 7 = 56$ is difficult for a student, give

**FIGURE 3.12** Verification with a mathematical balance.
the student six sheets of paper and a bold marker and have him or her write one large item on each sheet: 8, 7, 56, ×, +, and =. The student arranges the sheets in order to make true number sentences. See if the student can discover all eight number sentences that can be arranged horizontally.

Another way to help a student remember particular multiplication combinations is to have the student make drawings that represent those combinations. The author has done this by introducing measurement of area, specifically the area of a rectangle. Given a combination like $6 \times 7$ and a ruler, the student draws a 6 cm $\times$ 7 cm rectangle and determines the area to be 42 cm$^2$. This can be especially effective if the student has strong spatial intelligence.

Calculators can be used to help our students learn basic number combinations. For example, the constant function on a calculator can be used to help students generate products. For the products of 6 and numbers 2 through 9, children press $6$ $\div$ $6$ $\div$, and so on. Students will not lose count if they repeatedly press the equals key and say: 2 sixes is 12, 3 sixes is 18, and so on.

Individuals can also use calculators as they practice recalling number combinations. For example, a student says “six times seven” as he presses $6$ $\times$ $7$. Then, he puts his hand behind his back and says “equals 42” before he presses $42$. The student receives immediate confirmation that he was correct. If he was incorrect, he should repeat the complete procedure immediately.

Our students need to understand the operations of arithmetic; yet in time, they also need to be able to recall the basic number combinations. They need to make connections among combinations and acquire thinking strategies for finding missing numbers. Eventually, our students need practice—often in the form of games—to assure mastery of the basic number combinations.

**Using Alternative Algorithms**

If we are willing to accept the idea that there are many legitimate ways to subtract, divide, and so on, we could choose to introduce an algorithm that is fresh and new to those students who are experiencing difficulty. By doing so, we may circumvent any mindset of failure. We will also help students realize that there are many different ways of getting correct answers when computing.

Algorithms have differed historically, and they often differ among cultures today. Accordingly, numerous examples of alternative algorithms can be found in the literature of mathematics education. For example: Pearson describes pre-1900 algorithms, Berg describes five historical methods of multiplying whole numbers, Carroll and Porter describe a variety of alternative procedures for whole-number operations, as do Randolph and Sherman, and Philipp describes algorithms used by different cultures.

Three examples of alternative algorithms follow. With students who experience difficulty learning to compute, we may find that a low-stress procedure such as the first algorithm is learned with relative ease. Like traditional algorithms, low-stress procedures often can be taught as sensible records of manipulations with sticks, blocks, or number rods. The advantage of such algorithms is
that they separate fact recall from renaming and thereby place fewer demands for remembering on the student who is computing.

**Addition of Whole Numbers: Hutchings’s Low-Stress Method**

A.  
\[
\begin{array}{c}
4 \\
37 \\
\end{array}
\]

B.  
\[
\begin{array}{c}
6 \\
13 \\
\end{array}
\]

C.  
\[
\begin{array}{c}
5 \\
14 \\
9 \\
8 \\
2 \\
\end{array}
\]

D.  
\[
\begin{array}{c}
2 \\
6 \\
4 \\
7 \\
\end{array}
\]

\[
\begin{array}{c}
9 \\
9 \\
5 \\
15 \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
1 \\
1 \\
7 \\
\end{array}
\]

\[
3097
\]

As illustrated in examples A and B, sums for number combinations are written with small digits to the left and to the right instead of in the usual manner.

When a column is added, the one ten is ignored and the ones digit is added to the next number. In Example C, $5 + 9 = 14$, $4 + 8 = 12$, $2 + 4 = 6$, and $6 + 7 = 13$. The remaining three ones are recorded in the answer. The tens are counted and also recorded. Multi-column addition proceeds similarly, except the number of tens counted is recorded at the top of the next column.

Many students for whom regrouping in addition is difficult find this alternative rather easy to learn. It often brings success quickly, even with large examples. Other students, including those with perceptual problems, may be confused by the abundance of “crutches.” This is especially true when the examples are not written with large digits, which is the case with many published achievement tests.

**Subtraction of Whole Numbers: The Equal Additions (or European-Latino) Method**

A.  
\[
\begin{array}{c}
453 \\
18 \times 8 \\
\end{array}
\]

\[
5
\]

B.  
\[
\begin{array}{c}
453 \\
2 \times 8 \\
\end{array}
\]

\[
275
\]
A principle of compensation is applied: when equal quantities are added to both the minuend and the subtrahend, the difference remains the same. In this computation, ten is added to the sum, i.e., to the 3 in the ones place. To compensate for this addition, 10 is also added to the known addend; the 7 in the tens place is replaced with 8. Similarly, one hundred is added to the sum; i.e., the 5 in the tens place becomes a 15. To compensate, one hundred is added to the known addend; the 1 in the hundreds place is replaced with a 2.

**Subtraction of Rational Numbers:**

**The Equal Additions Method**

Problem: 

\[
\begin{array}{c}
7 \frac{4}{7} \\
-3 \frac{3}{4}
\end{array}
\]

\[
\begin{array}{c}
3 \frac{1}{2}
\end{array}
\]

add one

\[
\begin{array}{c}
\left( \frac{4}{4} \right)
\end{array}
\]

\[
\begin{array}{c}
\rightarrow 7 \frac{5}{4}
\end{array}
\]

add one

\[
\begin{array}{c}
\left( 1 \right)
\end{array}
\]

\[
\begin{array}{c}
\rightarrow 4 \frac{5}{4}
\end{array}
\]

\[
\begin{array}{c}
3 \frac{1}{4}
\end{array}
\]

Difference:

The principle of compensation is applied. One is added to both the minuend and the subtrahend in order to subtract easily.

**INVOLVING PEERS**

The teaching and learning of mathematics in our classrooms is greatly enhanced as we provide opportunities for our students to interact with one another. Peers can be involved in both assessment and instruction.

Chappuis and Stiggins rightly assert, “Classroom assessment that involves students in the process and focuses on increasing learning can motivate rather than merely measure students.” Sometimes students can be involved in peer assessment, during which students thoughtfully consider examples of work by other students, keeping specific criteria in mind. Students need to know what criteria are relevant to the situation at hand, and the criteria must make sense to them.

There is a very close relationship between peer assessment and self-assessment. When the same criteria are used for both self-assessment and peer assessment, students who assess peers also reflect on their own mathematical knowledge and skills. Students who respond to the work of other students (in reality or simulation) become more aware of their own knowledge and skills. Consider the sample in Figure 3.13.

For those students involved in peer assessment, there is value in the self-assessment that takes place; but there is also great value in students using what they know about mathematics to help others. At the same time, we as teachers learn much about both the student assessors and the peers whom they assess.

Sometimes a checklist can be used to guide peer assessment, for example the checklist in Figure 3.14. The assessor circles the Y (yes), the ? (I’m not sure), or the N (no).
Peer involvement in instruction can take the form of peer tutoring, or it can involve groups of students working together cooperatively. Peer tutoring, a strategy we may find helpful when we teach computation, occurs when students instruct one another by challenging, explaining, and demonstrating concepts and procedures. Often the tutor learns even more than the student being tutored.

If we plan carefully, peer tutoring can foster many of the emphases of *Principles and Standards for School Mathematics*: problems will be solved, assessment will be conducted, mathematical ideas will be discussed, reasoning will justify procedures, manipulatives and drawings will be used for representations, and connections will be made with prior learning. But these things will not happen unless we prepare carefully.

Barone and Taylor recommend at least a two-year age difference between tutors and their tutees. They suggest ways to implement peer tutoring with young children; nevertheless, many of their ideas apply to peer tutoring by older students as well. We need to select engaging instructional activities to be taught, then prepare tutors by teaching them the activity before they teach it to their peers. We can even have tutors teach the activity to other tutors while we observe, and have tutors switch roles. After tutoring sessions it is helpful to have both tutors and tutees write in their journals.

### Figure 3.13: Sample peer assessment.

Here is the way Gary multiplied. What could you tell Gary or show him to help him understand?

\[
\frac{5}{8} \times 3 = \frac{15}{24}
\]

### Figure 3.14: Sample checklist for peer assessment

<table>
<thead>
<tr>
<th></th>
<th>Name of student</th>
<th>Date on paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Are digits written in place-value columns?</td>
<td>Y ? N</td>
</tr>
<tr>
<td>2.</td>
<td>Can the student explain the procedure used?</td>
<td>Y ? N</td>
</tr>
<tr>
<td>3.</td>
<td>Will the procedure used always produce the correct answer?</td>
<td>Y ? N</td>
</tr>
<tr>
<td>4.</td>
<td>Did the student check the answer?</td>
<td>Y ? N</td>
</tr>
<tr>
<td>5.</td>
<td>Can the student describe a situation in which this computation could be used?</td>
<td>Y ? N</td>
</tr>
</tbody>
</table>

Comments

Signature of student evaluator

---


During classroom discussions about mathematical ideas, problem solving, and computation, students often learn from one another—especially English learners who may be in the class. These discussions also serve a diagnostic purpose, and we can modify instruction based on what we hear.71

Individual students learning to compute usually profit from involvement with cooperative groups of students while working on a problem or a task, especially when students work in a thoughtful and focused way. This is true for all students, including those who experience difficulty learning computational procedures. Students who work cooperatively to solve problems communicate mathematical ideas as they work; they both challenge and help one another.

If more than one limited-English-proficient student in your mathematics class is using the same native language, the use of this language often can be facilitated if those students learn cooperatively in small groups.72

Students working in cooperative groups need to be responsible for the learning of each person within the group (perhaps there is to be a team score). But we need to hold each individual accountable for his or her own learning. Individuals also need to be held accountable for contributing to the group.

Typically, groups should be structured, heterogeneous groups—often groups of four students. Cooperative groups can be structured by the tasks we give them. Tasks related to computational knowledge and skills include the following.

- Assign a set of computations to each student in a group of students. After individuals compute, they total their answers for the group, and compare that number with the specific number you provide.
- Discuss specific questions. For example, “What is the most important step in the procedure we studied? Why?”
- Solve a problem that involves deciding the appropriate method of computation, including procedures recently learned.
- Go over homework completed by individuals. Check answers with one another. Where answers differ, determine why, and determine which answer is correct. Students often assume they are correct and have to be convinced by others that they are not.
- Review for a test. Assign a sample test for the group to complete together.

When possible, structure the tasks so that students not only explain procedures to one another, but also so that each student makes sure other participants in the group understand and can do the procedures.73 If an activity involves a calculator, make sure students take turns using the calculator.

Whenever tasks involve manipulatives, we must remember that our students do not learn just by using manipulatives; they learn by thinking about what they are doing when they use manipulatives. They need to reflect on what they are doing with manipulatives and explain their reasoning to one another. It has been said that . . .

We should never tell a child what that child might be able to tell us. Similarly, we should never tell a child what some other child might be able to tell the child for us.
We need to carefully observe our students as they work. Perhaps groups are working on an assignment to create a task or problem that requires a particular computation, or each group has been given a problem for which they are to demonstrate two or more solutions. We must focus on their thinking as we observe them—not on their answers alone.

Sample activities for cooperative groups of students are described in Appendix D. The activities focus on computational procedures.

**USING PORTFOLIOS TO MONITOR PROGRESS DURING INSTRUCTION**

Do our students have individual goals for learning mathematics? Can each student point to growth? Portfolios showcase student work and provide especially helpful insights into student growth and accomplishment.

Students may already have work folders for filing completed work or work in progress, but each of our students also needs to have a mathematics portfolio into which selected examples of mathematics work are placed. It is appropriate to require certain papers to be included in their portfolios, but students should be encouraged to select many or even most of the items—items they especially value or believe show growth or creativity.

During instruction, we need to have our students reflect on work completed early in the year and compare it with work completed more recently. One of the main purposes for mathematics portfolios is to "help students develop better self-assessment skills and become less reliant on the grades we assign to their work." If we want students to understand why a certain computational procedure works, for instance, we may want to have students select the particular item from their work folder that best demonstrates their understanding of why the procedure works, and place it in their portfolio. Obviously, when we conference with parents, mathematics portfolios will be very useful.

Our purposes for having students develop mathematics portfolios will determine many of the types of items included. We may want to suggest that students include several of the following.

- Papers showing more than one way to solve a problem
- Papers showing why a traditional algorithm works
- Papers illustrating an alternative algorithm (or an algorithm invented by the student) and explaining why it works
- Papers that show how the student has corrected errors or misunderstandings
- Mathematical problems developed by the student; solutions may be included
- Papers displaying graphic organizers developed by the student; for example, a table of equivalent fractions
- Drawings of how manipulatives were used in solving a problem
- Individual or group reports of a project, such as a statistical survey with graphs
• Notes from an interview with someone about some aspect of mathematics
• Papers that show how mathematics is used in other subject areas
• Artwork involving geometric patterns or mathematical relationships
• Scale drawings
• Homework, especially solutions to nonroutine problems
• Performance assessment tasks given periodically
• Checklists completed by the teacher
• Mathematical autobiographies
• Writings describing how the student feels about mathematics class or “doing mathematics”
• Papers that respond to “What I Learned in Math Class Today”
• Papers that respond to “What I Learned from My Mistake,” written by students who have adopted error patterns in specific computation procedures
• Notes from the teacher describing evidence the student understood a particular mathematical principle
• Self-assessment sheets for groups and for individuals

Or we may want to have our students select from their work folders the five best items relative to a specific topic and place them in their mathematics portfolios along with a letter that explains why each was selected.

**Working with Parents**

Students need opportunities to show parents what they are learning.

Student-led conferences are not a substitute for teacher-parent conferences, but they are one way that parents can find out about their child’s experiences with mathematics at school. These conferences also help students understand which learnings are important and help students take responsibility for their own learning.

When student-led conferences are planned, students usually decide what part of their work the conference will focus on; they select work samples for their parents or guardians to see. During conferences, have students explain the criteria used for deciding that a particular work sample is well done. Encourage students to talk about their struggles as they learn mathematics—not just their successes. Student-led conferences need to be well planned, and students need to be prepared for their role in the conference.

Schools sometimes offer mini-courses for parents. These are often math-awareness workshops or math labs that focus on various topics students are studying. (Frequently, teachers learn as much from these workshops as the parents!)

With appropriate guidance, parents can stimulate their children’s interest in mathematics. Parents can show that the mathematics that children are learning is very much a part of their own daily life. They can describe how their jobs use mathematics (and how other jobs do as well), share their own strategies for estimating and solving problems, and show how they think through a specific problem.
We can help parents in many ways. They may:

- Need help realizing that their child is capable of learning mathematics, especially when they think, “I was never any good at math.”
- Need to understand the goals of mathematics instruction in your classroom.
- Need to understand how particular mathematics concepts and skills are sequenced.
- Need suggestions for getting their child organized, and for establishing a pattern of regular home study.
- Need to know how to link new knowledge and skills to what their child already knows and can do.
- Need to know strategies that will engage their children: for example, games that will provide practice their child is ready for.

Many parents need help asking appropriate questions while working with their children, questions like these:

- Did you estimate an answer?
- How does this relate to . . . ?
- Is there a drawing you could make that would help?
- How did you get to your answer?
- How would you convince me your answer makes sense?

Several publishers of textbooks have helps for parents. For example, Everyday Mathematics, developed by the University of Chicago School Mathematics Project, has a “Parent Homeroom” on their website; a glossary is provided, questions from parents are answered, and resources for parents are listed.

**GUIDING INSTRUCTION**

Teachers have found the following guidelines to be helpful. They provide a summary of principles to keep in mind when instructing students who are having difficulty learning to compute.

**Focus on the Student**

1. *Personalize instruction.* Even when students meet in groups for instruction, individuals must be assessed and programs must be planned for *individuals.* Some individual tutoring may be required.
2. *Believe the student is capable of learning.* A student who has met repeated failure needs to believe that she is a valued person and is capable of eventually acquiring the needed knowledge and skill. If you are to help, you must believe this, too.
3. *Make sure the student has the goals of instruction clearly in mind.* He needs to know the direction instruction is heading. He needs to know where it
will head eventually (“I’ll be able to subtract and get the right answers”) and where it is headed immediately (“I’ll soon be able to rename a number many different ways”).

4. **Encourage self-assessment.** From the beginning, involve students in the assessment process. Let them help set goals for instruction.

5. **Provide the student with a means to observe any progress.** Portfolios and journals, as well as charts and graphs, can serve this function.

6. **Ensure consistent expectations.** Make sure you and the student’s parents have the same expectations in regard to what the student will accomplish. People in the United States tend to assume that difficulties with learning result from a lack of ability; in many other countries, they are more likely to assume that difficulties are a result of insufficient effort. Be sure you and the student’s parents are together in regard to such expectations.

### Involve Parents

7. **Teach them strategies for engaging their children.** Many parents need to be helped if their children are going to be helped.

8. **Raise expectations if needed.** Some parents think, “I was never any good at math.” They need to believe their child can learn. Failure is not the norm.

### Teach Concepts and Skills

9. **Start with what the student knows and build on that knowledge.** She probably already knows more than you realize. Corrective instruction should build on a student’s strengths; it should consider what she is ready to learn. Typically, students need to understand subordinate mathematical concepts before they can be expected to integrate them into more complex ideas.

10. **Emphasize ideas that help a student organize what he learns.** Students often assume the concept or procedure they are learning applies only to the specific task they are involved in at the time. Connect new learnings with what the student already knows. When organized, new learnings can be more easily retrieved from memory as the need arises; also, they can be applied more readily in new contexts. Stress ideas such as multiple names for a number, commutativity, identity elements, and inverse relations.

11. **Stress the ability to estimate.** A student who makes errors in computation will become more accurate with the ability to determine the reasonableness of answers.

### Provide Instruction

12. **Base instruction on your diagnosis.** Take into account the patterns you observed while collecting data. What strengths can you build upon? Plan instruction that is data driven.
13. Use a great variety of instructional procedures and activities. Be sure to choose activities that differ from the way the student was previously taught, because he may associate previous instruction with fear and failure. Students develop ideas from experiences embodying the idea, and they typically perceive the concept as that which is common to all of the experiences. Therefore, variety is often necessary for adequate concept formation.

14. Involve students in higher-order thinking activities. If paper-and-pencil computation is your instructional goal, you may want to focus on a large problem or task that is challenging and interesting to the student. If instruction is to be fruitful, your goal must be to thoroughly engage the student; prompt him to think about what is happening during instruction.

15. Connect content to experiences out of school. A student who can tie what she is learning to experiences out of school is likely to be motivated to learn and be able to apply what she does learn.

16. Encourage the student to think out loud while working through a problem situation. Have him show how and explain why certain materials and procedures are being used. Speaking out loud often enables a student to focus more completely on the task at hand.

17. Ask leading questions that encourage reflection. Allow sufficient time for the student to reflect.

18. Let the student state his understanding of a concept or procedure in his own language. Do not always require the terminology of textbooks. It may be appropriate to say, for example, “Mathematicians have a special name for that idea, but I rather like your name for it!”

19. Sequence instruction in smaller amounts of content when needed. Some students having difficulty need smaller “chunks.” A large task may overwhelm such students. When instruction is based upon a sequence that leads to the larger task, these students can focus on more immediately attainable goals. Help them to see that immediate goals lead along a path going in the desired direction.

20. Move toward symbols gradually. Move from manipulatives to two-dimensional representations and visualizations to the use of symbols.

21. Emphasize careful penmanship and proper alignment of digits. A student must be able to read the work and tell the value assigned to each place where a digit is written. Columns can also be labeled if appropriate.

Use Concrete Materials

22. Let the student choose from materials available. Whenever possible, a student should be permitted to select a game or activity from materials that are available and that lead toward the goals of instruction. Identify activities for which the student has needed prerequisite skills and which lead to the goals of instruction; then let her have some choice in deciding what she will do.

23. Encourage a student to use aids as long as they are of value. Peer group pressure often keeps students from using aids even when the teacher
places aids on a table; the use of aids needs to be encouraged actively. Occasionally, a student needs to be prompted to try thinking a process through with just paper and pencil; but students often give up aids when they feel safe without them. After all, using aids is time consuming.

Provide Practice

24. Make sure a student understands the process before assigning practice. We have known for some time that, in general, drill reinforces and makes more efficient that which a student actually practices. In other words, if a student counts on his fingers to find a sum, drill will only tend to help him count on his fingers more efficiently. He may find sums more quickly, but he is apt to continue any immature procedure he is using. Avoid extensive use of practice activities at a time when they merely reinforce processes that are developmental. By looking for patterns of error and by conducting data-gathering interviews in an atmosphere in which incorrect responses are accepted, you can usually learn enough to decide if the student is ready for practice.

25. Include problems or puzzles to solve. Some of our practice activities need to be problems or puzzles that require students to compute while solving them. In this way, students not only gain practice with computational procedures, but they also apply skills they are learning. Examples of such activities are included in Appendix D; although the activities are designed for cooperative groups, several can be easily adapted for individual practice.

26. Select practice activities which provide immediate confirmation. When looking for games and drill activities to strengthen skills, choose activities that let students know immediately whether the answer is correct. Many games, manipulatives, and teacher-made devices provide such reinforcement. For example, magic squares are often used to provide practice with computation for all four operations—with fractions and decimals as well as with whole numbers.

27. Spread practice time over several short periods. Typically, a short series of examples (perhaps five to eight) is adequate to observe any error pattern. Longer series tend to reinforce erroneous procedures. If a correct procedure is being used, frequent practice with a limited number of examples is more fruitful than occasional practice with a large number of examples.

In Chapters 4 through 12 you will read specific suggestions for instruction. The lists of selected resources (pp. 232–243) point to additional ideas.

Reflecting on Instruction in Computation

Students need to learn to use different forms of computation. An estimate is often appropriate, but when an exact answer is needed, mental computation, calculators, or paper-and-pencil algorithms can be used. Our students need to be able to
do all of these calculations, and they need to be able to decide when each is most appropriate.

They also need to understand operations so they will know which operation is required in a particular problem situation. Many other concepts and principles need to be understood, including those involving numeration, equals, compensation, and properties of operations.

We can help our students understand these ideas by choosing concrete materials that are accurate mathematically and by planning engaging activities in which students reflect on what they do with the materials. Such activities are often springboards to acquisition of specialized mathematical vocabulary.

Ultimately, our students will need to both understand and recall basic number combinations. We need to teach thinking strategies before seeking mastery of addition and multiplication number combinations; when mastery is appropriate, games are often helpful. We also need to make sure that our students learn to use basic number combinations for addition and multiplication to answer subtraction and division questions.

We can enhance our mathematics instruction by having our students both talk and write mathematics, and by engaging students with one another. Graphic organizers often can help our students organize what they are learning and connect it to other concepts and to the world around them. Sometimes it is helpful if we teach an alternative algorithm.

The following chapters focus on paper-and-pencil computations. As you read, you will examine papers of students, identify error patterns, think about appropriate instruction, and receive feedback yourself.

REFERENCES

2. Ibid. p. 20.
5. Ibid. p. 78.
55. It is easier for students to count forward than backward for the unknown addend. This is actually a how-many-more meaning for subtraction and leads to the helpful practice of using related addition facts to answer subtraction questions.
56. Teaching distribution of multiplication over addition (multiplying “in parts”) can help students proceed independently when solving untaught or forgotten basic multiplication facts; they can derive facts from multiplication facts they remember. Show the multiplication by partitioning an array.